

AN EXTENSION PROCEDURE FOR THE CONSTRAINT EQUATIONS

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ABSTRACT. Let (g, k) be a solution to the maximal constraint equations of general relativity on the unit ball B_1 of \mathbb{R}^3 . We prove that if (g, k) is sufficiently close to the initial data for Minkowski space, then there exists an asymptotically flat solution (g', k') on \mathbb{R}^3 that extends (g, k) . Moreover, (g', k') depends continuously on (g, k) and has the same regularity.

Our proof uses a new method of solving the prescribed divergence equation for a tracefree symmetric 2-tensor, and a geometric variant of the conformal method to solve the prescribed scalar curvature equation for a metric. Both methods are based on the implicit function theorem and an expansion of tensors based on spherical harmonics. They are combined to define an iterative scheme that is shown to converge to a global solution (g', k') of the maximal constraint equations which extends (g, k) .

CONTENTS

1. Introduction	2
1.1. The Cauchy problem and the maximal constraint equations	2
1.2. The extension problem and the main theorem	3
1.3. Strategy of the proof of the main theorem	5
1.4. Acknowledgements	7
2. Preliminaries	7
2.1. Basic notation	7
2.2. The radial foliation of \mathbb{R}^3 by spheres S_r and tensor decomposition	8
2.3. Function spaces	10
2.4. Tensor spaces	13
2.5. Asymptotically flat initial data	15
2.6. L^2 -Hodge theory on S_r	16
2.7. The expansion of S_r -tangential tensors.	19
2.8. The implicit function theorem and Lipschitz estimates for operators	24
3. Precise statement of the main theorem	25
4. The divergence equation for k	26
4.1. Analysis of operators on $\mathcal{H}_{-1/2}^w$ -asymptotically flat metrics	26
4.2. Reduction to the Euclidean case	29
4.3. Surjectivity at the Euclidean metric	35
5. The prescribed scalar curvature equation for g	64

5.1. Scalar curvature and geometry of foliations	65
5.2. Reduction to the Euclidean case	69
5.3. Surjectivity at the Euclidean metric	74
6. Proof of the main Theorem 3.1	90
6.1. Setup of the iterative scheme	90
6.2. Convergence of the iterative scheme	91
Appendix A. The proof of Proposition 2.30	95
Appendix B. The proofs of Proposition 2.34 and Lemma 2.35	96
Appendix C. Elliptic operators on weighted Sobolev spaces	99
C.1. Weak formulation of boundary value problems in weighted spaces	99
C.2. Elliptic estimates in \overline{H}_δ^1	101
C.3. Higher elliptic regularity in $H_\delta^w(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{H}_\delta^1$ and \overline{H}_δ^w	104
C.4. An elliptic estimate in L^2	106
C.5. Estimates to apply Lemma C.2	109
References	110

1. INTRODUCTION

1.1. The Cauchy problem and the maximal constraint equations. The Einstein vacuum equations on a Lorentzian 4-manifold $(\mathcal{M}, \mathbf{g})$ are given by

$$\text{Ric}(\mathbf{g}) = 0,$$

where Ric denotes the Ricci tensor of \mathbf{g} . Initial data for the Cauchy problem is given by a triple (Σ, g, k) , where (Σ, g) is a complete Riemannian 3-manifold and k a symmetric 2-tensor on Σ satisfying the *constraint equations* on Σ

$$\begin{aligned} R(g) &= |k|_g^2 - (\text{tr}_g k)^2, \\ \text{div}_g k &= d(\text{tr}_g k). \end{aligned} \tag{1.1}$$

Here $R(g)$ denotes the scalar curvature of g , d is the exterior derivative and

$$|k|_g^2 := g^{ij} g^{lm} k_{il} k_{jm}, \quad \text{tr}_g k := g^{ij} k_{ij}, \quad (\text{div}_g k)_l := g^{ij} \nabla_i k_{jl},$$

where ∇ denotes the covariant derivative on (Σ, g) and we use, as in the rest of this paper, the Einstein summation convention.

Let $(\mathcal{M}, \mathbf{g})$ be the solution of the Einstein vacuum equations corresponding to initial data (Σ, g, k) . Then $\Sigma \subset (\mathcal{M}, \mathbf{g})$ is a space-like Cauchy hypersurface with induced metric g and second fundamental form k . See for example [31] for details.

The trivial solution to the Einstein vacuum equations is the Minkowski spacetime. Its initial data is given by

$$(\Sigma, g, k) = (\mathbb{R}^3, e, 0),$$

where e denotes the Euclidean metric.

In this work, we consider initial data that satisfies two further properties.

- The initial data is *asymptotically flat*, which means

$$g \rightarrow e, k \rightarrow 0$$

as $|x| \rightarrow \infty$ on Σ . For a more precise definition, see Definition 2.20. Such initial data corresponds to the description of isolated gravitational systems, see for example [31].

- We assume that Σ is *maximal*, that is,

$$\mathrm{tr}_g k = 0.$$

This assumption is sufficiently general, see for example [4].

By the second assumption, the constraint equations (1.1) reduce to the *maximal constraint equations*

$$\begin{aligned} R(g) &= |k|^2, \\ \mathrm{div}_g k &= 0, \\ \mathrm{tr}_g k &= 0. \end{aligned} \tag{1.2}$$

1.2. The extension problem and the main theorem. The maximal constraint equations are an under-determined geometric non-linear elliptic system of partial differential equations. In this paper, we are interested in the following problem.

Extension problem. *Given initial data (g, k) on the unit ball $B_1 \subset \mathbb{R}^3$, does there exist a regular asymptotically flat initial data set (g', k') on \mathbb{R}^3 that isometrically contains (g, k) and continuously depends on it?*

This problem has received considerable attention in the literature. It appears for example

- when analysing the space of solutions to the maximal constraint equations, see for example [5] [29] [26] [18] [28] [25],
- when considering the rigidity of the equations, as in the celebrated gluing construction [12] [13], see also [10] [11],
- in the context of Bartnik's conjecture [6] [7], see for example [17] [22] [27] [2] [3]
- in the proof of the bounded L^2 curvature theorem [20], where it is used to reduce the local existence for the Cauchy problem of general relativity to the small data case, see for example Section 2.3 in that paper.

Our main motivation to consider the extension problem is to prove a *localised version* of the bounded L^2 curvature theorem of [20], see the forthcoming publication [15].

In this paper, we resolve the extension problem for small data. The next theorem is a rough version of our main result, see Theorem 3.1 for a precise formulation.

Theorem 1.1 (Main theorem, version 1). *Let (g, k) be a solution on the unit ball $B_1 \subset \mathbb{R}^3$ of the maximal constraint equations*

$$\begin{aligned} R(g) &= |k|_g^2, \\ \operatorname{div}_g k &= 0, \\ \operatorname{tr}_g k &= 0. \end{aligned}$$

If (g, k) is sufficiently close to $(e, 0)$ in a suitable topology, then there exists asymptotically flat (g', k') of the same regularity as (g, k) such that

$$(g', k')|_{B_1} = (g, k),$$

and solving the maximal constraint equations on \mathbb{R}^3 ,

$$\begin{aligned} R(g') &= |k'|_{g'}^2, \\ \operatorname{div}_{g'} k' &= 0, \\ \operatorname{tr}_{g'} k' &= 0. \end{aligned}$$

Moreover, (g', k') depends continuously on (g, k) .

Comments on the result.

- (1) The novelty of our result lies in the following facts.
 - Compared to the gluing construction [12] [13], it does not need a gluing region. This feature is crucial for localising the bounded L^2 curvature theorem [20], see the forthcoming [15].
 - The extension results in [29] [26] [28] [25] lose regularity across the boundary of the domain by using a parabolic equation to solve the prescribed scalar curvature equation. Our result, on the other hand, uses a geometric perturbation argument at the Euclidean metric that preserves regularity.
- (2) The closeness of (g, k) to $(e, 0)$ is measured in the topology corresponding to the space

$$(g, k) \in \mathcal{H}^w(B_1) \times \mathcal{H}^{w-1}(B_1),$$

where $\mathcal{H}^w(B_1)$ denotes a Sobolev space of tensors over B_1 corresponding to w derivatives in L^2 . We note that Theorem 1.1 applies for integers $w \geq 2$ (see the precise version Theorem 3.1) and therefore in particular for weak regularity. In view of the scaling of (1.2), we expect Theorem 1.1 to hold also for real numbers w in the range $w > 3/2$.

- (3) Theorem 1.1 completes the proof of the reduction step to small data in the proof of the bounded L^2 curvature theorem [20], see Section 2.3 in that paper.

- (4) The methods used in the proof of Theorem 1.1 could be relevant to other problems such as, for example, solving the divergence equation in context of the Maxwell-Klein-Gordon and Euler equations, see for example [19] [23].

1.3. Strategy of the proof of the main theorem. In this section we sketch the proof of Theorem 1.1. The idea is to set up an iterative scheme consisting of pairs $((g_i, k_i))_{i \geq 1}$ that extend (g, k) from B_1 to \mathbb{R}^3 . In general, the (g_i, k_i) do not solve the maximal constraints (1.2) on \mathbb{R}^3 . However, by a fixpoint argument, we show that the sequence converges to a solution (g', k') as $i \rightarrow \infty$.

More precisely, let (g, k) be small given initial data on B_1 , and assume we have already obtained (g_i, k_i) for some $i \geq 1$. We construct the next pair (g_{i+1}, k_{i+1}) by the following two steps.

- **Step A.** Given (g_i, k_i) on \mathbb{R}^3 , construct g_{i+1} on \mathbb{R}^3 such that

$$\begin{aligned} g_{i+1}|_{B_1} &= g, \\ R(g_{i+1}) &= |k_i|_{g_i}^2. \end{aligned}$$

- **Step B.** Given g_{i+1} on \mathbb{R}^3 , construct k_{i+1} on \mathbb{R}^3 such that

$$\begin{aligned} k_{i+1}|_{B_1} &= k, \\ \operatorname{div}_{g_{i+1}} k_{i+1} &= 0, \\ \operatorname{tr}_{g_{i+1}} k_{i+1} &= 0. \end{aligned}$$

Step A and B rely on Theorems 1.3 and 1.2, respectively, to be introduced now.

Theorem 1.2 (Extension of divergence-free tracefree symmetric 2-tensors, version 1). *Let g' be an asymptotically flat Riemannian metric on \mathbb{R}^3 and k a symmetric 2-tensor on B_1 solving*

$$\begin{aligned} \operatorname{div}_{g'} k &= 0, \\ \operatorname{tr}_{g'} k &= 0. \end{aligned} \tag{1.3}$$

If g' and k are sufficiently close to e and 0, respectively, in a suitable topology, then there exists an asymptotically flat symmetric 2-tensor k' on \mathbb{R}^3 that extends k , that is,

$$k'|_{B_1} = k$$

and solves on \mathbb{R}^3

$$\begin{aligned} \operatorname{div}_{g'} k' &= 0, \\ \operatorname{tr}_{g'} k' &= 0. \end{aligned} \tag{1.4}$$

Moreover, k' depends continuously on k .

Theorem 1.3 (Metric extension theorem, version 1). *Let g be a Riemannian metric on B_1 and R a scalar function on \mathbb{R}^3 such that*

$$R|_{B_1} = R(g),$$

where $R(g)$ denotes the scalar curvature of g . If g and R are sufficiently close to e and 0, respectively, then there exists an asymptotically flat Riemannian metric g' on \mathbb{R}^3 such that

$$g'|_{B_1} = g,$$

and such that its scalar curvature on \mathbb{R}^3 is given by

$$R(g') = R.$$

Moreover, g' depends continuously on g and R .

Precise versions of the above are stated in Theorems 4.1 and 5.1, respectively. Both Theorems 1.2 and 1.3 are proved by the Implicit Function Theorem and showing the surjectivity of a linearisation of the corresponding operators at the Euclidean metric e . Concerning Theorem 1.2, we show in Section 4.3 that the operator

$$k \mapsto \rho := \operatorname{div}_e(\widehat{k}^e)$$

is surjective. Here, for any symmetric 2-tensor V , we denote its tracefree part with respect to the Euclidean metric e by

$$\widehat{V}^e := V - \frac{1}{3}\operatorname{tr}_e(V)e.$$

Concerning Theorem 1.3, we show in Section 5.3 that the linearisation of the scalar curvature with respect to a suitable geometric variation is surjective.

The two proofs of surjectivity at the Euclidean metric use, among others, the following mathematical tools.

- (1) In Section 2.2, we decompose tensors with respect to the foliation of \mathbb{R}^3 by spheres

$$S_r = \{|x| = r\}, r > 0.$$

- (2) In Section 2.7, relying on spherical harmonics, we define complete orthonormal bases of the space of $L^2(S_r)$ -integrable functions, vectorfields and symmetric trace-free 2-tensors on S_r . We call these bases Hodge-Fourier bases. Projecting onto these bases allows us to split up the linearised operators into radial ODEs and elliptic systems on S_r and $\mathbb{R}^3 \setminus \overline{B_1}$.
- (3) In order to force the continuity of the normal derivative at the boundary of B_1 , it is necessary to control the Dirichlet-to-Neumann maps associated to the elliptic systems on $\mathbb{R}^3 \setminus \overline{B_1}$. This is achieved in particular by exploiting the underdetermined character of the constraint equations.

The rest of the paper is organised as follows. In Section 2, we introduce the notation and weighted Sobolev spaces and bases of functions and tensors. In Section 3 we state a precise version of Theorem 1.1. In Section 4, we first reduce the proof of Theorem 1.2 to the surjectivity at the Euclidean space which is then proved in Section 4.3. Similarly, in Section 5, we first reduce the proof of Theorem 1.3 to the surjectivity at the Euclidean space which is then proved in Section 5.3. In Section 6, we set up the iterative scheme and prove Theorem 1.1. In Appendix A, we prove the completeness of the bases of tensors defined in Section 2. Two lemmas of Section 2 are proved in Appendix B. In Appendix C, derive elliptic estimates in weighted Sobolev spaces.

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2. PRELIMINARIES

2.1. Basic notation. In this work, miniscule Latin indices range over $a, b, c, d, i, j = 1, 2, 3$, majuscule Latin indices over $A, B, C, D = 1, 2$ and $n \in \mathbb{N}$. The index pairs (lm) take as values integers $l \geq 0, m \in \{-l, \dots, l\}$. We apply the Einstein summation convention. The notation $A \lesssim B$ means $A \leq cB$ where $c > 0$ is a numerical constant that does not depend on A, B .

An open subset $\Omega \subset \mathbb{R}^3$ is called a *domain* if it is connected and its boundary $\partial\Omega := \overline{\Omega} \setminus \overset{\circ}{\Omega}$ is smooth. Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a fixed smooth transition function such that

$$\chi(x) = \begin{cases} 0 & \text{for } x \leq 1/10, \\ 1 & \text{for } x \geq 1. \end{cases} \quad (2.1)$$

We work in a fixed Cartesian coordinate system (x^1, x^2, x^3) of \mathbb{R}^3 . Consequently, given a n -tensor T , we can equivalently denote it by its coordinate components $T_{i_1 \dots i_n}$.

Let e denote the Euclidean metric on \mathbb{R}^3 with components

$$e_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let g be a Riemannian metric and V a symmetric 2-tensor. Let the divergence, the symmetrized curl, the trace and the tracefree part of V with respect to g be

$$\begin{aligned} (\operatorname{div}_g V)_j &:= \nabla^i V_{ij}, \\ (\operatorname{curl}_g V)_{ij} &:= \frac{1}{2}(\epsilon_i^{ab} \nabla_a V_{bj} + \epsilon_j^{ab} \nabla_a V_{bi}), \\ \operatorname{tr}_g V &:= g^{ab} V_{ab}, \\ \widehat{V}^g &:= V - \frac{1}{3} \operatorname{tr}_g(V) g, \end{aligned}$$

where ∇ denotes the covariant derivative and ϵ the volume form of g .

2.2. The radial foliation of \mathbb{R}^3 by spheres S_r and tensor decomposition. Let

$$(r, \theta^1, \theta^2) \in [0, \infty) \times [0, \pi) \times [0, 2\pi)$$

denote the standard polar coordinates on \mathbb{R}^3 . By definition they are related to the Cartesian coordinates (x^1, x^2, x^3) by

$$\begin{aligned} x^1 &= r \sin \theta^1 \cos \theta^2, \\ x^2 &= r \sin \theta^1 \sin \theta^2, \\ x^3 &= r \cos \theta^1. \end{aligned}$$

The coordinate spheres and balls of radius $r_0 > 0$ centered at the origin are respectively defined as

$$\begin{aligned} S_{r_0} &:= \{x \in \mathbb{R}^3 : r(x) = r_0\}, \\ B_{r_0} &:= \{x \in \mathbb{R}^3 : r(x) < r_0\}. \end{aligned}$$

In standard polar coordinates, the Euclidean metric is given by

$$e = dr^2 + r^2((d\theta^1)^2 + \sin^2 \theta^1 (d\theta^2)^2).$$

Let the induced metric on $S_r \subset (\mathbb{R}^3, e)$ be denoted by

$$\overset{\circ}{\gamma} := r^2((d\theta^1)^2 + \sin^2 \theta^1 (d\theta^2)^2).$$

When integrating over $(S_r, \overset{\circ}{\gamma})$, we do not write out the standard volume element.

The *standard polar frame* on $\mathbb{R}^3 \setminus \{0\}$ is defined as

$$\left\{ \partial_r, e_1 := \frac{1}{r} \partial_{\theta^1}, e_2 := \frac{1}{r \sin \theta^1} \partial_{\theta^2} \right\}, \quad (2.2)$$

where $\partial_r, \partial_{\theta^1}, \partial_{\theta^2}$ are the coordinate vectorfields in the coordinate system (r, θ^1, θ^2) , respectively.

Every Riemannian metric g on $\mathbb{R}^3 \setminus \{0\}$ can be uniquely written as

$$g = a^2 dr^2 + \gamma_{AB} (\beta^A dr + d\theta^A) (\beta^B dr + d\theta^B), \quad (2.3)$$

where

- $a(x) > 0$ for all $x \in \mathbb{R}^3 \setminus \{0\}$ is the positive lapse function,
- γ is the Riemannian metric induced by g on S_r , $r > 0$,
- β is the S_r -tangent shift vector.

The a, γ, β are called the *polar components* of g .

The following lemma is proved by direct calculation.

Lemma 2.1. *Let g be a Riemannian metric given on $\mathbb{R}^3 \setminus \{0\}$,*

$$g = a^2 dr^2 + \gamma_{AB} (\beta^A dr + d\theta^A) (\beta^B dr + d\theta^B).$$

Then the following holds for any $r > 0$.

- *The outward-pointing¹ unit normal N to S_r with respect to g is given by*

$$N = \frac{1}{a} \partial_r - \frac{1}{a} \beta.$$

- *The second fundamental form² Θ of S_r with respect to g equals in any coordinates on S_r , $A, B = 1, 2$,*

$$\Theta_{AB} = -\frac{1}{2a} \partial_r (\gamma_{AB}) + \frac{1}{2a} (\mathcal{L}_\beta \gamma)_{AB}, \quad (2.4)$$

where \mathcal{L} denotes the Lie derivative on S_r .

Remark 2.2. *The polar components of the Euclidean metric e are*

$$a = 1, \quad \beta = 0,$$

$$\gamma_{AB} = \overset{\circ}{\gamma}_{AB} = \begin{cases} r^2 & \text{if } A = B = 1, \\ r^2 \sin^2 \theta^1 & \text{if } A = B = 2, \\ 0 & \text{if } A \neq B. \end{cases}$$

Furthermore, $N = \partial_r$ and

$$\mathfrak{t} := \overset{\circ}{\gamma}^{AB} \Theta_{AB} = -\frac{2}{r}, \quad |\Theta|_{\overset{\circ}{\gamma}}^2 := \overset{\circ}{\gamma}^{AC} \overset{\circ}{\gamma}^{BD} \Theta_{AB} \Theta_{CD} = \frac{2}{r^2}.$$

More generally, we now decompose vectorfields and symmetric 2-tensors on $\mathbb{R}^3 \setminus \overline{B_1}$ with respect to the foliation of \mathbb{R}^3 by spheres S_r . Given a vectorfield X , decompose it into

- the scalar function X_N ,
- the S_r -tangent vectorfield $X_A = X_A$,

¹That is, pointing into the unbounded connected component of $\mathbb{R}^3 \setminus S_r$.

²Here we use the sign convention that $\Theta(X, Y) := -g(X, \nabla_Y N)$.

where $A = 1, 2$ denote components in any frame on S_r .

Given a symmetric 2-tensor V , decompose it into

- the scalar function V_{NN} ,
- the S_r -tangent vectorfield $(\mathcal{V}_N)_A := V_{NA}$,
- the S_r -tangent 2-tensor $\mathcal{V}_{AB} := V_{AB}$,

where $A, B = 1, 2$ denote components in any frame on S_r .

Definition 2.3. Let X be a S_r -tangent vectorfield and V a S_r -tangent symmetric 2-tensor on $\mathbb{R}^3 \setminus \{0\}$. Define the S_r -tangential vectorfield $\nabla_N X$ and symmetric 2-tensor $\nabla_N V$, respectively, by

$$\begin{aligned} (\nabla_N X)_a &:= (\Pi_{TS_r})_a^c \nabla_N X_c, \\ (\nabla_N V)_{ab} &:= (\Pi_{TS_r})_a^c (\Pi_{TS_r})_b^d \nabla_N V_{cd}, \end{aligned}$$

where $a, b = 1, 2, 3$ and

$$(\Pi_{TS_r})_{ij} := \delta_{ij} - N_i N_j$$

denotes the projection onto TS_r , where here δ_{ij} is the Kronecker symbol.

2.3. Function spaces.

Definition 2.4 (Sobolev spaces). Let $\Omega \subset \mathbb{R}^3$ be a domain and $w \geq 0$ integer. Let $H^w(\Omega)$ denote the standard Sobolev space

$$H^w(\Omega) := \left\{ f \in L^2(\Omega) : \sum_{|\alpha| \leq w} \|\partial^\alpha f\|_{L^2(\Omega)} < \infty \right\}.$$

Here $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ is a multi-index and

$$|\alpha| := \alpha_1 + \alpha_2 + \alpha_3, \quad \partial^\alpha := \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}.$$

Definition 2.5. Let $\Omega \subset \mathbb{R}^3$ be a domain and $w \geq 0$ an integer. Define $H_{loc}^w(\Omega)$ as

$$H_{loc}^w(\Omega) := \bigcap_{\Omega' \subset \subset \Omega} H^w(\Omega'),$$

where $\Omega' \subset \subset \Omega$ denotes all domains Ω' such that $\overline{\Omega'}$ is compact and $\overline{\Omega'} \subset \Omega$.

See for example [1] for properties of the above function spaces. Our analysis of the constraint equations is set in the following weighted Sobolev spaces.

Definition 2.6 (Weighted Sobolev spaces). Let $\Omega \subset \mathbb{R}^3$ be a domain, $w \geq 0$ an integer and $\delta \in \mathbb{R}$. Let

$$H_\delta^w(\Omega) := \left\{ f \in H_{loc}^w(\Omega) : \sum_{|\beta| \leq w} \|(1+r)^{-\delta-3/2+|\beta|} \partial^\beta f\|_{L^2(\Omega)} < +\infty \right\}.$$

Furthermore, define

$$H_\delta^w := H_\delta^w(\mathbb{R}^3).$$

For $w \geq 0$ integer and $\delta \in \mathbb{R}$, $H_\delta^w(\Omega)$ is a Hilbert space. The next three lemmas follow from Lemmas 2.1, 2.4 and 2.5 in [21].

Lemma 2.7. *Let $\delta, \delta_1, \delta_2 \in \mathbb{R}$, $w, w_1, w_2 \geq 0$ integers and f a scalar function on \mathbb{R}^3 . The following holds.*

- If $w \geq 1$ and $f \in H_\delta^w$, then $\partial f \in H_{\delta-1}^{w-1}$.
- If $0 \leq w_1 \leq w_2$ and $\delta_1 \leq \delta_2$, then $H_{\delta_1}^{w_1} \subset H_{\delta_2}^{w_2}$.
- For $w \geq 2$, the space H_δ^w continuously embeds into

$$\left\{ f \in L_{loc}^\infty(\mathbb{R}^3) : \sum_{|\beta| \leq w-2} \sup_{x \in \mathbb{R}^3} (1+r)^{-\delta+|\beta|} |\partial^\beta f| < \infty \right\}.$$

Lemma 2.8. *Let $w, w_1, w_2 \geq 0$ be integers such that $w \leq \min(w_1, w_2)$ and $w \leq w_1 + w_2 - 2$. Let further $\delta_1, \delta_2 \in \mathbb{R}$. Then for any $(u, v) \in H_{\delta_1}^{w_1} \times H_{\delta_2}^{w_2}$, it holds that $uv \in H_{\delta_1+\delta_2}^w$ and*

$$\|uv\|_{H_{\delta_1+\delta_2}^w} \lesssim \|u\|_{H_{\delta_1}^{w_1}} \|v\|_{H_{\delta_2}^{w_2}}.$$

Lemma 2.9. *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function. Let the scalar function $u \in H_{\delta_1}^{w_1}$ for an integer $w_1 \geq 2$ and $\delta_1 < 0$, and $v \in H_{\delta_2}^{w_2}$ for an integer $0 \leq w_2 \leq w_1$ and $\delta_2 \in \mathbb{R}$. The following holds.*

- (1) *There exists a constant $C = C(\|u\|_{H_{\delta_1}^{w_1}}, F) > 0$ such that*

$$\|F(u)v\|_{H_{\delta_2}^{w_2}} \leq C\|v\|_{H_{\delta_2}^{w_2}}.$$

- (2) *For any sequence $(u_n, v_n)_{n \in \mathbb{N}}$ such that $(u_n, v_n) \rightarrow (u, v)$ in $H_{\delta_1}^{w_1} \times H_{\delta_2}^{w_2}$ as $n \rightarrow \infty$, it holds that*

$$F(u_n)v_n \rightarrow F(u)v \text{ in } H_{\delta_2}^{w_2} \text{ as } n \rightarrow \infty,$$

in other words, the map $(u, v) \mapsto F(u)v$ is continuous.

The following two corollaries are used in Sections 4 and 5.

Corollary 2.10. *For $w \geq 2, \delta < 0$, the space H_δ^w forms an algebra.*

The proof of Corollary 2.10 follows by Lemma 2.8 and is left to the reader.

Corollary 2.11. *Let the scalar function $F : \mathbb{R} \rightarrow \mathbb{R}$ be smooth in an open neighbourhood of 0. Let $w_1 \geq 2$ and $0 \leq w_2 \leq w_1$ be integers, $\delta_1 < 0, \delta_2 \in \mathbb{R}$. There is a constant $\varepsilon > 0$ such that the mapping*

$$(u, v) \mapsto F(u)v$$

is smooth from $B_\varepsilon(0) \times H_{\delta_2}^{w_2}$ to $H_{\delta_2}^{w_2}$, where

$$B_\varepsilon(0) := \left\{ u : \|u\|_{H_{\delta_1}^{w_1}} < \varepsilon \right\} \subset H_{\delta_1}^{w_1}.$$

Moreover, for all functions

$$u, \tilde{u} \in B_\varepsilon(0) \subset H_{\delta_1}^{w_1}, \quad v \in H_{\delta_2}^{w_2}$$

it holds that

$$\|(F(u) - F(\tilde{u}))v\|_{H_{\delta_2}^{w_2}} \lesssim \|u - \tilde{u}\|_{H_{\delta_1}^{w_1}} \|v\|_{H_{\delta_2}^{w_2}}, \quad (2.5)$$

$$\|F(u) - F(\tilde{u})\|_{H_{\delta_1}^{w_1}} \lesssim \|u - \tilde{u}\|_{H_{\delta_1}^{w_1}} \quad (2.6)$$

Proof. The existence of $\varepsilon > 0$ such that the mapping is smooth in $u \in B_\varepsilon(0) \subset H_{\delta_1}^{w_1}$ follows by applying the L^∞ -estimate of Lemma 2.7 and Lemmas 2.8 and 2.9, Corollary 2.10 to derivatives of $F(u)v$ with respect to u, v .

It remains to prove the Lipschitz estimates (2.5) and (2.6). Indeed, for $\varepsilon > 0$ sufficiently small, by Lemmas 2.8 and 2.9 and the fact that $\delta_1 < 0$,

$$\begin{aligned} \|(F(u) - F(\tilde{u}))v\|_{H_{\delta_2}^{w_2}} &\leq \int_0^1 \|DF|_{su+(1-s)\tilde{u}}(u - \tilde{u})v\|_{H_{\delta_2}^{w_2}} ds \\ &\leq \int_0^1 \|DF|_{su+(1-s)\tilde{u}}(u - \tilde{u})v\|_{H_{\delta_2+\delta_1}^{w_2}} ds \\ &\leq \int_0^1 \|DF|_{su+(1-s)\tilde{u}}(u - \tilde{u})\|_{H_{\delta_1}^{w_1}} ds \|v\|_{H_{\delta_2}^{w_2}} \\ &\lesssim \|u - \tilde{u}\|_{H_{\delta_1}^{w_1}} \|v\|_{H_{\delta_2}^{w_2}}, \end{aligned}$$

where we used that DF is smooth on $B_\varepsilon(0)$ for $\varepsilon > 0$ small. The proof of (2.6) is similar and left to the reader. This concludes the proof of Corollary 2.11. \square

Definition 2.12. Let $\Omega \subset \mathbb{R}^3$ be a domain, $w \geq 0$ an integer and $\delta \in \mathbb{R}$. Define $\overline{H}_\delta^w(\Omega)$ to be the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H_\delta^w(\Omega)}$. Further, define

$$\overline{H}_\delta^w := \overline{H}_\delta^w(\mathbb{R}^3 \setminus \overline{B_1}).$$

The following useful characterisation of \overline{H}_δ^w is left to the reader, see for example Exercise 3 of Section 4.5 in [30].

Proposition 2.13. Let $w \geq 2$ be an integer, $\delta \in \mathbb{R}$. Let $u \in H_\delta^w(\mathbb{R}^3 \setminus \overline{B_1})$. The following are equivalent.

- (1) The trivial extension of u to B_1 is regular, that is $\overline{u} \in H_\delta^w$, where

$$\overline{u} = \begin{cases} u(x) & \text{if } x \in \mathbb{R}^3 \setminus \overline{B_1}, \\ 0 & \text{if } x \in \overline{B_1}. \end{cases}$$

(2) For $l = 0, \dots, w-1$, it holds that in the trace sense,

$$\partial_r^l u|_{r=1} = 0.$$

(3) It holds that $u \in \overline{H}_\delta^w$.

In dimension 1, the following Sobolev embedding holds. This is similar to Lemma 2.7 and its proof is left to the reader.

Lemma 2.14. *Let $\delta \in \mathbb{R}$. Let $u : (1, \infty) \rightarrow \mathbb{R}$ be a scalar function. If*

$$\int_1^\infty (1+r)^{-2\delta-1} u^2(r) dr, \int_1^\infty (1+r)^{-2\delta+1} (\partial_r u)^2(r) dr, \int_1^\infty (1+r)^{-2\delta+3} (\partial_r^2 u)^2(r) dr < +\infty,$$

then $u, \partial_r u \in C^0((1, \infty))$ and

$$\sup_{r \in (1, \infty)} (1+r)^{-\delta} u(r), \sup_{r \in (1, \infty)} (1+r)^{-\delta+1} \partial_r u(r) < +\infty.$$

For functions on $(S_r, \overset{\circ}{\gamma})$, we define the following norm.

Definition 2.15. *Let $w \geq 0$ be an integer. Let f be a function on S_r for some $r > 0$. Then*

$$\|f\|_{H^w(S_r)}^2 := \sum_{0 \leq n \leq w} \int_{S_r} |\nabla^n f|_{\overset{\circ}{\gamma}}^2,$$

where ∇ denotes the covariant derivative on $(S_r, \overset{\circ}{\gamma})$ and

$$|\nabla^n f|_{\overset{\circ}{\gamma}}^2 = \overset{\circ}{\gamma}^{A_1 B_1} \dots \overset{\circ}{\gamma}^{A_n B_n} \nabla_{A_1} \dots \nabla_{A_n} f \nabla_{B_1} \dots \nabla_{B_n} f,$$

see Definition 2.16. Denote further $H^0(S_r) = L^2(S_r)$.

2.4. Tensor spaces. More generally, we now define tensor spaces on \mathbb{R}^3 .

Definition 2.16. *Given an n -tensor T and a Riemannian metric g , let*

$$|T|_g^2 := g^{i_1 j_1} \dots g^{i_n j_n} T_{i_1 \dots i_n} T_{j_1 \dots j_n}.$$

In case of the Euclidean metric e , for an n -tensor T ,

$$|T|_e^2 = \sum_{i_1, \dots, i_n=1}^3 |T_{i_1 \dots i_n}|^2.$$

The norm of a tensor is defined as follows.

Definition 2.17 (Tensor norms). *Let $\Omega \subset \mathbb{R}^3$ be a domain. Let $n \geq 1$ and $w \geq 0$ be integers. For an n -tensor T on Ω , define its $\mathcal{H}^w(\Omega)$ -norm by*

$$\|T\|_{\mathcal{H}^w(\Omega)}^2 := \sum_{|\alpha| \leq w} \int_{\Omega} |\partial^\alpha T|_e^2,$$

where $(\partial^\alpha T)_{i_1 \dots i_n} = \partial^\alpha (T_{i_1 \dots i_n})$. We write $T \in \mathcal{H}^w(\Omega)$ if this norm is finite. We similarly define tensors in $\mathcal{H}_{loc}^w(\Omega)$, $\mathcal{H}_\delta^w(\Omega)$, $\overline{\mathcal{H}}_\delta^w(\Omega)$, \mathcal{H}_δ^w and $\overline{\mathcal{H}}_\delta^w$.

We define tensor norms on $(S_r, \overset{\circ}{\gamma})$ as follows.

Definition 2.18. Let $w \geq 0$ be an integer. Let T be a S_r -tangent tensor on $(S_r, \overset{\circ}{\gamma})$ for some $r > 0$. Then

$$\|T\|_{\mathcal{H}^w(S_r)}^2 := \sum_{0 \leq n \leq w} \int_{S_r} |\nabla^n T|_{\overset{\circ}{\gamma}}^2,$$

where ∇ denotes the covariant derivative on $(S_r, \overset{\circ}{\gamma})$. We say that tensors in $\mathcal{H}^0(S_r)$ are L^2 -integrable.

The next lemma is practical for calculations.

Lemma 2.19. Let $w \geq 0$ be an integer. Let X be a vectorfield and V a symmetric 2-tensor on $\mathbb{R}^3 \setminus \overline{B_1}$. Then

$$\begin{aligned} \|X\|_{\mathcal{H}^w(\mathbb{R}^3 \setminus \overline{B_1})}^2 &= \|X_N\|_{H^w(\mathbb{R}^3 \setminus \overline{B_1})}^2 + \|X\|_{\mathcal{H}^w(\mathbb{R}^3 \setminus \overline{B_1})}^2 \\ &\approx \|X_N\|_{H^w(\mathbb{R}^3 \setminus \overline{B_1})}^2 + \sum_{n_1+n_2 \leq w} \int_1^\infty \int_{S_r} |\nabla^{n_1} \nabla^{n_2} X|_{\overset{\circ}{\gamma}}^2 dr, \\ \|V\|_{\mathcal{H}^w(\mathbb{R}^3 \setminus \overline{B_1})}^2 &= \|V_{NN}\|_{H^w(\mathbb{R}^3 \setminus \overline{B_1})}^2 + \|\mathcal{V}_N\|_{\mathcal{H}^w(\mathbb{R}^3 \setminus \overline{B_1})}^2 + \|\mathcal{V}\|_{\mathcal{H}^w(\mathbb{R}^3 \setminus \overline{B_1})}^2 \\ &\approx \|V_{NN}\|_{H^w(\mathbb{R}^3 \setminus \overline{B_1})}^2 + \sum_{n_1+n_2 \leq w} \int_1^\infty \int_{S_r} |\nabla^{n_1} \nabla^{n_2} \mathcal{V}_N|_{\overset{\circ}{\gamma}}^2 dr \\ &\quad + \sum_{n_1+n_2 \leq w} \int_1^\infty \int_{S_r} |\nabla^{n_1} \nabla^{n_2} \mathcal{V}|_{\overset{\circ}{\gamma}}^2 dr, \end{aligned}$$

where here ∇ denotes the tangential gradient and ∇_N was defined in Definition 2.3. Analogously for $\mathcal{H}_{loc}^w(\mathbb{R}^3 \setminus \overline{B_1})$, $\mathcal{H}_\delta^w(\mathbb{R}^3 \setminus \overline{B_1})$, $\overline{\mathcal{H}}_\delta^w$.

The proof of the above lemma is left to the reader. It follows by using that with the radial tensor decomposition of Section 2.2, it holds that for a vectorfield X and a symmetric 2-tensor V ,

$$\begin{aligned} |X|_e^2 &= X_N^2 + |X|_{\overset{\circ}{\gamma}}^2, \\ |V|_e^2 &= V_{NN}^2 + |\mathcal{V}_N|_{\overset{\circ}{\gamma}}^2 + |\mathcal{V}|_{\overset{\circ}{\gamma}}^2. \end{aligned}$$

2.5. Asymptotically flat initial data. The following definition is standard, see for example [21].

Definition 2.20 (Asymptotically flat initial data). *Let $w \geq 2$ be an integer. Let $g \in \mathcal{H}_{loc}^w(\mathbb{R}^3)$ be a Riemannian metric and $k \in \mathcal{H}_{loc}^{w-1}(\mathbb{R}^3)$ a symmetric 2-tensor on \mathbb{R}^3 . The metric g is called $\mathcal{H}_{-1/2}^w$ -asymptotically flat if*

$$g - e \in \mathcal{H}_{-1/2}^w,$$

where e denotes the Euclidean metric on \mathbb{R}^3 . The pair (g, k) is called $\mathcal{H}_{-1/2}^w$ -asymptotically flat if

$$g - e \in \mathcal{H}_{-1/2}^w, \quad k \in \mathcal{H}_{-3/2}^{w-1}, \quad (2.7)$$

Similarly, a metric g on $\mathbb{R}^3 \setminus \overline{B_1}$ is called $\mathcal{H}_{-1/2}^w$ -asymptotically flat if

$$g - e \in \mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1}).$$

Remark 2.21. The norms associated to (2.7) are explicitly

$$\begin{aligned} \sum_{i,j=1}^3 \sum_{|\alpha| \leq w} \|(1+r)^{-1+|\alpha|} \partial^\alpha (g_{ij} - e_{ij})\|_{L^2(\mathbb{R}^3)} &< +\infty, \\ \sum_{i,j=1}^3 \sum_{|\alpha| \leq w-1} \|(1+r)^{|\alpha|} \partial^\alpha k_{ij}\|_{L^2(\mathbb{R}^3)} &< +\infty. \end{aligned}$$

The next lemma allows us to directly work with the polar components of an $\mathcal{H}_{-1/2}^w$ -asymptotically flat metric.

Lemma 2.22. *Let $w \geq 2$ be an integer. There exists a universal $\varepsilon > 0$ small such that the following holds.*

(1) *Let g be an $\mathcal{H}_{-1/2}^w$ -asymptotically flat Riemannian metric on $\mathbb{R}^3 \setminus \overline{B_1}$ such that*

$$\|g - e\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} < \varepsilon,$$

and denote its polar components on $\mathbb{R}^3 \setminus \overline{B_1}$ by

$$g = a^2 dr^2 + \gamma_{AB} (\beta^A dr + d\theta^A) (\beta^B dr + d\theta^B).$$

It holds that

$$a^2 - 1 \in H_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1}), \beta, \gamma - \overset{\circ}{\gamma} \in \mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})$$

with the estimate

$$\|a^2 - 1\|_{H_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} + \|\beta\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} + \|\gamma - \overset{\circ}{\gamma}\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|g - e\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})}.$$

- (2) Let a be a positive scalar function, β a S_r -tangent vectorfield and a γ Riemannian metric on S_r on $\mathbb{R}^3 \setminus \overline{B_1}$ such that

$$\|a^2 - 1\|_{H_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} + \|\beta\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} + \|\gamma - \overset{\circ}{\gamma}\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} < \varepsilon.$$

The symmetric 2-tensor g defined on $\mathbb{R}^3 \setminus \overline{B_1}$ by

$$g = a^2 dr^2 + \gamma_{AB} (\beta^A dr + d\theta^A) (\beta^B dr + d\theta^B).$$

is an $\mathcal{H}_{-1/2}^w$ -asymptotically flat Riemannian metric and bounded by

$$\begin{aligned} \|g - e\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|a^2 - 1\|_{H_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} + \|\beta\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \\ &\quad + \|\gamma - \overset{\circ}{\gamma}\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})}. \end{aligned}$$

Proof. The proof follows from Lemma 2.19 applied to $g - e$. Indeed, in case of a metric,

$$g_{NN} = a^2, (\not{g}_N)_A = \gamma_{AB} \beta^B, \not{g} = \gamma.$$

By Lemma 2.19 and Remark 2.2, we can bound

$$\|a^2 - 1\|_{H_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})}, \|\gamma(\beta, \cdot)\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})}, \|\gamma - \overset{\circ}{\gamma}\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|g - e\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})}.$$

Moreover, for $\|\gamma - \overset{\circ}{\gamma}\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})}$ sufficiently small, γ is invertible, and so the vectorfield β is also controlled by Lemma 2.9. This proves part (1) of Lemma 2.22.

Part (2) is demonstrated similarly by Lemma 2.19 and left to the reader. This finishes the proof of Lemma 2.22. \square

2.6. L^2 -Hodge theory on S_r . In this section, we recall basic Hodge theory on Euclidean spheres $(S_r, \overset{\circ}{\gamma})$, $r > 0$. This is a special case of the Hodge theory on Riemannian 2-spheres in [9]. All tensors are assumed to be S_r -tangent. Let

- ∇ denote the covariant derivative on $(S_r, \overset{\circ}{\gamma})$.
- $\not{\epsilon}$ denote the volume element on $(S_r, \overset{\circ}{\gamma})$.
- $\not{\Delta} := \overset{\circ}{\gamma}^{AB} \nabla_A \nabla_B$ denote the Laplace-Beltrami operator³ on $(S_r, \overset{\circ}{\gamma})$.
- the divergence and curl of a vectorfield X be defined as

$$\begin{aligned} \text{div} \xi &:= \nabla_A X^A, \\ \text{curl} \xi &:= \not{\epsilon}_{AB} \nabla^A X^B. \end{aligned}$$

- the divergence and trace of a symmetric 2-tensor V_{AB} be defined as

$$\begin{aligned} (\text{div} V)_B &:= \nabla^A V_{AB}, \\ \text{tr} V &:= \overset{\circ}{\gamma}^{AB} V_{AB}. \end{aligned}$$

³Here we follow the convention that Laplacians have negative eigenvalues.

- for a vectorfield X the symmetric 2-tensor $\nabla \widehat{\otimes} X$ be defined as

$$(\nabla \widehat{\otimes} X)_{AB} := \nabla_A X_B + \nabla_B X_A - (\text{div} X) \overset{\circ}{\gamma}_{AB}.$$

- for two vectorfields X, Y the symmetric tracefree 2-tensor $X \widehat{\otimes} Y$ be defined as

$$(X \widehat{\otimes} Y)_{AB} := X_A Y_B + X_B Y_A - \overset{\circ}{\gamma}(X, Y) \overset{\circ}{\gamma}_{AB}.$$

- the left Hodge dual of a vectorfield X be defined as

$$*X_A := \epsilon_{AB} X^B.$$

- the left Hodge dual of a symmetric tracefree 2-tensor V be defined as

$$*V_{AB} := \epsilon_{AC} V^C_B.$$

- the modulus of an n -tensor V be defined as

$$|V|^2 := \overset{\circ}{\gamma}^{A_1 B_1} \cdots \overset{\circ}{\gamma}^{A_n B_n} V_{A_1 \cdots A_n} V_{B_1 \cdots B_n}.$$

We note that for a vectorfield X and a symmetric tracefree 2-tensor V ,

$$*(X) := -X, *(V) := -V. \quad (2.8)$$

Introduce two Hodge systems on $(S_r, \overset{\circ}{\gamma})$ as follows. Let X be a vectorfield on S_r that verifies

$$\begin{aligned} \text{div} X &= f, \\ \text{curl} X &= f_*, \end{aligned} \quad (\text{H1})$$

where f, f_* are scalar functions on S_r .

Let V be a tracefree symmetric 2-tensor on S_r that verifies

$$\text{div} V = F, \quad (\text{H2})$$

where F is a 1-form on S_r .

The following is the Euclidean version of Proposition 2.2.1 in [9].

Proposition 2.23 (Ellipticity of Hodge systems). *The following holds.*

- Assume that the vectorfield X is a solution of \mathbf{H}_1 . Then

$$\int_{S_r} \left(|\nabla X|^2 + \frac{1}{r^2} |X|^2 \right) = \int_{S_r} (|f|^2 + |f_*|^2).$$

- Assume that the symmetric tracefree 2-tensor V is a solution of \mathbf{H}_2 . Then

$$\int_{S_r} \left(|\nabla V|^2 + \frac{2}{r^2} |V|^2 \right) = 2 \int_{S_r} |F|^2.$$

Furthermore, the next higher regularity estimates hold.

Proposition 2.24 (Higher regularity for Hodge systems on S_r). *Let $w \geq 1$ be an integer. The following holds.*

- Assume that the vectorfield X is a solution of \mathbf{H}_1 for $f, f_* \in H^{w-1}(S_r)$. Then

$$\sum_{0 \leq n \leq w} \int_{S_r} |r^n \nabla^n X|^2 \lesssim \sum_{0 \leq n \leq w-1} \int_{S_r} r^2 (|r^n \nabla^n f|^2 + |r^n \nabla^n f_*|^2).$$

- Assume that the symmetric tracefree 2-tensor V is a solution of \mathbf{H}_2 for $F \in \mathcal{H}^{w-1}(S_r)$. Then

$$\sum_{0 \leq n \leq w} \int_{S_r} |r^n \nabla^n V|^2 \lesssim \sum_{0 \leq n \leq w-1} \int_{S_r} r^2 |r^n \nabla^n F|^2.$$

Proof. We only give a sketch of the proof, because it follows from Lemmas 2.2.2 and 2.2.3 in [9] and the fact that we work on the round sphere $(S_r, \overset{\circ}{\gamma})$. The proof is by induction on w . The case $w = 1$ is Proposition 2.23. The induction step $w \rightarrow w + 1$ follows by showing that the symmetrized derivative of a totally symmetric tensor ξ ,

$$\tilde{D}\xi_{A_1 A_2 \dots A_{k+1} B} := \frac{1}{k+2} \left(\nabla_B \xi_{A_1 \dots A_{k+1}} + \sum_{i=1}^{k+1} \nabla_{A_i} \xi_{A_1 \dots B \dots A_{k+1}} \right)$$

satisfies a Hodge system whose source terms can be controlled⁴ in lower order norms of ξ . Lemma 2.2.2 in [9] shows the ellipticity of this Hodge system. Generally on S_r , the symmetrized derivative and the curl of a tensor control the full covariant derivative, see Chapter 2 of [9]. The curl is estimated via the Hodge system by the induction assumption so that the full control of $\nabla \xi$ follows. This finishes the proof of Proposition 2.24. \square

The following relations are from Chapter 2 in [9]:

Lemma 2.25. *Let \mathcal{P}_1 be the operator that takes a vectorfield X on S_r into the pair of functions $(\text{div} X, \text{curl} X)$. The L^2 -adjoint of \mathcal{P}_1 is the operator \mathcal{P}_1^* which takes pairs of functions (f, f_*) into vectorfields on S_r given by*

$$\mathcal{P}_1^*(f, f_*) = -\nabla^A f + \epsilon^{AB} \nabla_B f_*.$$

Let \mathcal{P}_2 be the operator that takes a symmetric tracefree 2-tensor X into the 1-form $\text{div} X$. The L^2 -adjoint of \mathcal{P}_2 is \mathcal{P}_2^ which takes 1-forms F into symmetric tracefree 2-tensors given by*

$$\mathcal{P}_2^* F = -\frac{1}{2} (\nabla \hat{\otimes} F)_{AB},$$

where we recall that

$$(\nabla \hat{\otimes} F)_{AB} := \nabla_A F_B + \nabla_B F_A - (\text{div} F) \overset{\circ}{\gamma}_{AB}.$$

⁴Thereby it is used that in the Euclidean case, the Gauss curvature $K = 1/r^2$ is spherically symmetric, so in particular $\nabla K = 0$.

The following relations hold.

$$\begin{aligned}\mathcal{P}_1\mathcal{P}_1^* &= -\Delta, \\ \mathcal{P}_2\mathcal{P}_2^* &= -\frac{1}{2}\Delta - \frac{1}{2}\frac{1}{r^2}, \\ \mathcal{P}_1^*\mathcal{P}_1 &= -\Delta + \frac{1}{r^2}, \\ \mathcal{P}_2^*\mathcal{P}_2 &= -\frac{1}{2}\Delta + \frac{1}{r^2}.\end{aligned}$$

Remark 2.26. By the above, the kernel of \mathcal{P}_2^* can be identified with the conformal Killing vectorfields on $(S_r, \overset{\circ}{\gamma})$. This implies that the image of \mathcal{P}_2 is $L^2(S_r)$ -orthogonal to the conformal Killing vectorfields of $(S_r, \overset{\circ}{\gamma})$.

2.7. The expansion of S_r -tangential tensors. In Sections 4.3 and 5.3, we analyse Hodge systems on Euclidean spheres. The main technical tools for this analysis are the bases of tensors defined here in the following. In this section, all differential operators are on Euclidean spheres $(S_r, \overset{\circ}{\gamma})$ and all tensors are S_r -tangent.

- Real spherical harmonics: For $r > 0$, let

$$\left\{ Y^{(lm)}(r, \theta, \phi) : l \geq 0, m \in \{-l, \dots, l\} \right\}$$

denote the set of normalised real spherical harmonics on S_r . In particular, for each $l \geq 0, m \in \{-l, \dots, l\}$ they solve

$$\Delta Y^{(lm)} = -\frac{l(l+1)}{r^2} Y^{(lm)}. \quad (2.9)$$

The next lemma is standard, see for example [14].

Lemma 2.27. For each $r > 0$, the set

$$\left\{ Y^{(lm)}(r) : l \geq 0, m \in \{-l, \dots, l\} \right\}$$

forms a complete orthonormal basis of $L^2(S_r)$ -integrable scalar functions on S_r .

- Vector spherical harmonics: For $r > 0$, let the vectorfields $E^{(lm)}, H^{(lm)}$ on S_r be defined for $l \geq 1, m \in \{-l, \dots, l\}$ by

$$\begin{aligned}E^{(lm)}(r) &:= \frac{r}{\sqrt{l(l+1)}} \mathcal{P}_1^*(Y^{(lm)}, 0), \\ H^{(lm)}(r) &:= \frac{r}{\sqrt{l(l+1)}} \mathcal{P}_1^*(0, Y^{(lm)}),\end{aligned} \quad (2.10)$$

where \mathcal{P}_1^* is given in Lemma 2.25.

- 2-covariant spherical harmonics: For $r > 0$, let the tracefree symmetric 2-tensors $\psi^{(lm)}, \phi^{(lm)}$ on S_r be defined for $l \geq 2, m \in \{-l, \dots, l\}$ by

$$\begin{aligned}\psi_{AB}^{(lm)}(r) &:= \frac{r}{\sqrt{\frac{1}{2}l(l+1)-1}} \mathcal{P}_2^*(E^{(lm)}), \\ \phi_{AB}^{(lm)}(r) &:= \frac{r}{\sqrt{\frac{1}{2}l(l+1)-1}} \mathcal{P}_2^*(H^{(lm)}),\end{aligned}\tag{2.11}$$

where \mathcal{P}_2^* is given in Lemma 2.25.

Remark 2.28. *The tensors defined in (2.10) and (2.11) are spherical harmonics in the sense that by Lemma 2.25,*

- for $l \geq 1, m \in \{-l, \dots, l\}$,

$$\begin{aligned}\Delta E^{(lm)} &= \frac{1-l(l+1)}{r^2} E^{(lm)}, \\ \Delta H^{(lm)} &= \frac{1-l(l+1)}{r^2} H^{(lm)}.\end{aligned}$$

- for $l \geq 2, m \in \{-l, \dots, l\}$,

$$\begin{aligned}\Delta \psi^{(lm)} &= \frac{4-l(l+1)}{r^2} \psi^{(lm)}, \\ \Delta \phi^{(lm)} &= \frac{4-l(l+1)}{r^2} \phi^{(lm)}.\end{aligned}$$

The next proposition shows that these sets of tensors form complete orthonormal bases. First, we introduce some notation.

Definition 2.29. *Let $r > 0$. Let f be a scalar function, X a vectorfield and V a symmetric tracefree 2-tensor on S_r . Define then*

- for $l \geq 0$: $f^{(lm)}(r) := \int_{S_r} Y^{(lm)} f$,
- for $l \geq 1$: $X_E^{(lm)}(r) := \int_{S_r} X \cdot E^{(lm)}$, $X_H^{(lm)}(r) := \int_{S_r} X \cdot H^{(lm)}$,
- for $l \geq 2$: $V_\psi^{(lm)}(r) := \int_{S_r} V \cdot \psi^{(lm)}$, $V_\phi^{(lm)}(r) := \int_{S_r} V \cdot \phi^{(lm)}$,

where \cdot denotes the contraction of tensors with respect to $\overset{\circ}{\gamma}$.

Proposition 2.30. *For all $r > 0$, the set*

$$\left\{ E^{(lm)}(r), H^{(lm)}(r) : l \geq 1, m \in \{-l, \dots, l\} \right\}$$

forms a complete orthonormal basis of the space of L^2 -integrable vectorfields on S_r . For all $r > 0$, the set

$$\left\{ \psi^{(lm)}(r), \phi^{(lm)}(r) : l \geq 2, m \in \{-l, \dots, l\} \right\}$$

forms a complete orthonormal basis of the set of L^2 -integrable tracefree symmetric 2-tensors on S_r . Moreover,

- for any scalar function $f \in L^2(S_r)$,

$$\|f\|_{L^2(S_r)}^2 = \sum_{l \geq 0} \sum_{m=-l}^l (f^{(lm)})^2,$$

- for any S_r -tangent vectorfield $X \in \mathcal{H}^0(S_r)$,

$$\|X\|_{\mathcal{H}^0(S_r)}^2 = \sum_{l \geq 1} \sum_{m=-l}^l \left((X_E^{(lm)})^2 + (X_H^{(lm)})^2 \right),$$

- for any S_r -tangent symmetric tracefree 2-tensor $V \in \mathcal{H}^0(S_r)$,

$$\|V\|_{\mathcal{H}^0(S_r)}^2 = \sum_{l \geq 2} \sum_{m=-l}^l \left((V_\psi^{(lm)})^2 + (V_\phi^{(lm)})^2 \right).$$

A proof is given in Appendix A.

Remark 2.31. For all $r > 0$, the vectorfields with $l = 1$,

$$\left\{ E^{(1m)}(r), H^{(1m)}(r) : m \in \{-1, 0, 1\} \right\},$$

form an orthonormal basis of the six-dimensional space of conformal Killing fields on $(S_r, \overset{\circ}{\gamma})$.

The next expansion notation is used throughout Sections 4.3, 5.3 and Appendix C.

Definition 2.32. Let $f \in L^2(S_r)$ be a scalar function, $X \in \mathcal{H}^0(S_r)$ a S_r -tangent vectorfield and $V \in \mathcal{H}^0(S_r)$ a S_r -tangent tracefree symmetric 2-tensor. Denote

$$\begin{aligned} f &= \underbrace{f^{(00)} Y^{(00)}}_{:=f^{[0]}} + \underbrace{\sum_{m=-1}^1 f^{(1m)} Y^{(1m)}}_{:=f^{[1]}} + \underbrace{\sum_{l \geq 2} \sum_{m=-l}^l f^{(lm)} Y^{(lm)}}_{:=f^{[\geq 2]}}, \\ X &= \underbrace{\sum_{m=-1}^1 X_E^{(1m)} E^{(1m)}}_{:=X_E^{[1]}} + \underbrace{\sum_{m=-1}^1 X_H^{(1m)} H^{(1m)}}_{:=X_H^{[1]}} + \underbrace{\sum_{l \geq 2} \sum_{m=-l}^l X_E^{(lm)} E^{(lm)}}_{:=X_E^{[\geq 2]}} + \underbrace{\sum_{l \geq 2} \sum_{m=-l}^l X_H^{(lm)} H^{(lm)}}_{:=X_H^{[\geq 2]}}, \\ V &= \underbrace{\sum_{l \geq 2} \sum_{m=-l}^l V_\psi^{(lm)} \psi^{(lm)}}_{:=V_\psi} + \underbrace{\sum_{l \geq 2} \sum_{m=-l}^l V_\phi^{(lm)} \phi^{(lm)}}_{:=V_\phi}, \end{aligned}$$

and let $X^{[1]} = X_E^{[1]} + X_H^{[1]}$, $X^{[\geq 2]} = X_E^{[\geq 2]} + X_H^{[\geq 2]}$.

We have the following identities.

Lemma 2.33 (Hodge-Fourier calculus). *Let $f \in L^2(S_r)$ be a scalar function, $X \in \mathcal{H}^0(S_r)$ a vectorfield and $V \in \mathcal{H}^0(S_r)$ a symmetric tracefree 2-tensor. It holds that*

- for $l \geq 1, m \in \{-l, \dots, l\}$,

$$\begin{aligned} -(\nabla f)_E^{(lm)} &= \frac{\sqrt{l(l+1)}}{r} f^{(lm)}, & -(\nabla f)_H^{(lm)} &= 0, \\ (\operatorname{div} X)^{(lm)} &= \frac{\sqrt{l(l+1)}}{r} X_E^{(lm)}, & (\operatorname{curl} X)^{(lm)} &= \frac{\sqrt{l(l+1)}}{r} X_H^{(lm)}, \end{aligned}$$

- for $l \geq 2, m \in \{-l, \dots, l\}$,

$$\begin{aligned} -\frac{1}{2} (\nabla \widehat{\otimes} X)_\psi^{(lm)} &= \frac{\sqrt{\frac{1}{2}l(l+1)-1}}{r} X_E^{(lm)}, & -\frac{1}{2} (\nabla \widehat{\otimes} X)_\phi^{(lm)} &= \frac{\sqrt{\frac{1}{2}l(l+1)-1}}{r} X_H^{(lm)}, \\ (\operatorname{div} V)_E^{(lm)} &= \frac{\sqrt{\frac{1}{2}l(l+1)-1}}{r} V_\psi^{(lm)}, & (\operatorname{div} V)_H^{(lm)} &= \frac{\sqrt{\frac{1}{2}l(l+1)-1}}{r} V_\phi^{(lm)}. \end{aligned}$$

The proof of this lemma follows by (2.10), (2.11), Lemma 2.25 and integration by parts. Details are left to the reader.

The next three results are handy for the estimates in Section 4.3 and 5.3.

Proposition 2.34. *Let u be a scalar function and X a vectorfield on S_r for some $r > 0$. For all integers $w \geq 0$,*

$$\begin{aligned} \|\nabla^w u\|_{\mathcal{H}^0(S_r)}^2 &\approx \sum_{l \geq 0} \sum_{m=-l}^l \left(\frac{l(l+1)}{r^2} \right)^w (u^{(lm)})^2, \\ \|\nabla^w X\|_{\mathcal{H}^0(S_r)}^2 &\approx \sum_{l \geq 1} \sum_{m=-l}^l \left(\frac{l(l+1)-1}{r^2} \right)^w \left((X_E^{(lm)})^2 + (X_H^{(lm)})^2 \right). \end{aligned}$$

A proof is provided in Appendix B.

Lemma 2.35. *Let $w \geq 0$ be an integer and $r > 0$. The following holds.*

- (1) *Let f be a scalar function. Then, for $l \geq 0, m \in \{-l, \dots, l\}$,*

$$\left| (\partial_r^w f)^{(lm)}(r) \right| \lesssim \sum_{n=0}^w \frac{|\partial_r^{w-n}(f^{(lm)})|}{r^n}. \quad (2.12)$$

(2) Let X be a S_r -tangent vectorfield. Then, for $l \geq 1, m \in \{-l, \dots, l\}$,

$$\left| (\nabla_N^w X)_E^{(lm)}(r) \right| \lesssim \sum_{n=0}^w \frac{\left| \partial_r^{w-n} \left(X_E^{(lm)} \right) \right|}{r^n}, \quad (2.13)$$

$$\left| (\nabla_N^w X)_H^{(lm)}(r) \right| \lesssim \sum_{n=0}^w \frac{\left| \partial_r^{w-n} \left(X_H^{(lm)} \right) \right|}{r^n}. \quad (2.14)$$

Moreover,

$$\begin{aligned} \partial_r (r \operatorname{div} X) &= r \operatorname{div} (\nabla_N X), \\ \partial_r (r \operatorname{curl} X) &= r \operatorname{curl} (\nabla_N X). \end{aligned}$$

(3) Let V be a S_r -tangent tracefree symmetric 2-tensor. Then, for $l \geq 2, m \in \{-l, \dots, l\}$,

$$\left| (\nabla_N^n V)_\psi^{(lm)}(r) \right| \lesssim \sum_{n=0}^w \frac{\left| \partial_r^{w-n} \left(V_\psi^{(lm)} \right) \right|}{r^n}, \quad (2.15)$$

$$\left| (\nabla_N^w V)_\phi^{(lm)}(r) \right| \lesssim \sum_{n=0}^w \frac{\left| \partial_r^{w-n} \left(V_\phi^{(lm)} \right) \right|}{r^n}. \quad (2.16)$$

Moreover,

$$\nabla_N (r \operatorname{div} V) = r \operatorname{div} (\nabla_N V).$$

A proof is provided in Appendix B.

Lemma 2.36. Let $w \geq 0$ be an integer. Let $f^{[\geq 1]}, f^{*[\geq 1]} \in H^w(S_r)$ be scalar functions and $X^{[\geq 2]} \in \mathcal{H}^w(S_r)$ a vectorfield. Then the inverse maps

$$\begin{aligned} \mathcal{P}_1^{-1} : (f^{[\geq 1]}, f^{*[\geq 1]}) &\mapsto \mathcal{P}_1^{-1}(f^{[\geq 1]}, f^{*[\geq 1]}), \\ \mathcal{P}_2^{-1} : (X^{[\geq 2]}) &\mapsto \mathcal{P}_2^{-1}(X^{[\geq 2]}), \end{aligned}$$

into vectorfields and tracefree symmetric 2-tensors, defined such that $\mathcal{P}_1^{-1}(f^{[\geq 1]}, f^{*[\geq 1]})$ and $\mathcal{P}_2^{-1}(X^{[\geq 2]})$ respectively solve on S_r

$$\begin{cases} \operatorname{div} \mathcal{P}_1^{-1}(f^{[\geq 1]}, f^{*[\geq 1]}) = f^{[\geq 1]}, \\ \operatorname{curl} \mathcal{P}_1^{-1}(f^{[\geq 1]}, f^{*[\geq 1]}) = f^{*[\geq 1]}, \\ \operatorname{div} \mathcal{P}_2^{-1}(X^{[\geq 2]}) = X^{[\geq 2]}, \end{cases}$$

are well-defined and continuous maps into $\mathcal{H}^{w+1}(S_r)$, respectively. Moreover, for any scalar function $f^{[\geq 1]}$ on S_r ,

$$\begin{aligned} (\mathcal{P}_1^{-1}(f^{[\geq 1]}, 0))_H^{[\geq 1]} &= 0, & (\mathcal{P}_1^{-1}(0, f^{[\geq 1]}))_E^{[\geq 1]} &= 0, \\ (\mathcal{P}_2^{-1}(E^{(lm)}))_\phi^{[\geq 1]} &= 0, & (\mathcal{P}_2^{-1}(H^{(lm)}))_\psi^{[\geq 1]} &= 0. \end{aligned}$$

The above lemma is a consequence of Lemma 2.25, Remarks 2.26 and 2.31 and Propositions 2.23 and 2.24 and its proof is left to the reader.

2.8. The implicit function theorem and Lipschitz estimates for operators. For completeness we state the standard Implicit Function Theorem that is used in Sections 4.2 and 5.2, see for example Theorem 2.5.7 in [24] for a proof.

Theorem 2.37. *Let X, Y, Z be Hilbert spaces. Let $U \subset X, V \subset Y$ be open subsets and $\mathcal{F} : U \times V \rightarrow Z$ be a C^r -mapping, $r \geq 1$. For some $x_0 \in U, y_0 \in V$ assume that the linearization in the first argument*

$$D_1\mathcal{F}|_{(x_0, y_0)} : X \rightarrow Z$$

is an isomorphism. Then there are open neighbourhoods $V_0 \subset V$ of y_0 and $W_0 \subset Z$ of $\mathcal{F}(x_0, y_0)$ and a unique C^r -mapping $\mathcal{G} : V_0 \times W_0 \rightarrow U$ such that for all $(y, z) \in V_0 \times W_0$,

$$\mathcal{F}(\mathcal{G}(y, z), y) = z.$$

We also need the following lemma.

Lemma 2.38. *Let X, Y, Z be Hilbert spaces. Let $T : X \times Y \rightarrow Z$ be a C^r -mapping for $r \geq 2$ in an open neighbourhood of $(0, 0) \in X \times Y$ such that for all $x \in X$,*

$$T(x, 0) = 0.$$

There exists an $\varepsilon > 0$ such that the following holds.

- *For $(x, y) \in B_\varepsilon(0) \times B_\varepsilon(0) \subset X \times Y$ it holds that*

$$\|T(x, y)\|_Z \lesssim \|y\|_Y.$$

- *For $x, x' \in B_\varepsilon(0) \subset X$ and $y \in B_\varepsilon(0) \subset Y$ it holds that*

$$\|T(x, y) - T(x', y)\|_Z \lesssim \|x - x'\|_X \|y\|_Y.$$

Proof. First,

$$\begin{aligned} \|T(x, y)\|_Z &= \|T(x, y) - T(x, 0)\|_Z \\ &\leq \int_0^1 \|D_2 T|_{(x, ty)}(y)\|_Z dt \\ &\lesssim \|y\|_Y, \end{aligned}$$

where we used that for $\varepsilon > 0$ small, the operator T is C^1 on $B_\varepsilon(0) \times B_\varepsilon(0) \subset X \times Y$. Second,

$$\begin{aligned}
\|T(x, y) - T(x', y)\|_Z &= \|T(x, y) - T(x, 0) - T(x', y) + T(x', 0)\|_Z \\
&\leq \int_0^1 \|D_2 T|_{(x, ty)}(y) - D_2 T|_{(x', ty)}(y)\|_Z dt \\
&\leq \int_0^1 \left(\int_0^1 \|D_1 D_2 T|_{(sx+(1-s)x', ty)}(y)(x - x')\|_Z ds \right) dt \\
&\lesssim \|x - x'\|_X \|y\|_Y,
\end{aligned} \tag{2.17}$$

where we used that for $\varepsilon > 0$ small, the operator T is C^2 on $B_\varepsilon(0) \times B_\varepsilon(0) \subset X \times Y$. This finishes the proof of Lemma 2.38. \square

3. PRECISE STATEMENT OF THE MAIN THEOREM

We are now in the position to state the precise version of our main theorem.

Theorem 3.1 (Main theorem, version 2). *Let $w \geq 2$ be an integer. Let g be a Riemannian metric and k a symmetric 2-tensor on B_1 that solve*

$$\begin{aligned}
R(g) &= |k|_g^2, \\
\operatorname{div}_g k &= 0, \\
\operatorname{tr}_g k &= 0.
\end{aligned}$$

There exists a universal constant $\varepsilon > 0$ such that if

$$\|(g - e, k)\|_{\mathcal{H}^w(B_1) \times \mathcal{H}^{w-1}(B_1)} < \varepsilon,$$

where e denotes the Euclidean metric, then there is an $\mathcal{H}_{-1/2}^w$ -asymptotically flat Riemannian metric g' on \mathbb{R}^3 and a symmetric 2-tensor $k' \in \mathcal{H}_{-3/2}^{w-1}$ such that

$$(g', k')|_{B_1} = (g, k)$$

and such that on \mathbb{R}^3

$$\begin{aligned}
R(g') &= |k'|_{g'}^2, \\
\operatorname{div}_{g'} k' &= 0, \\
\operatorname{tr}_{g'} k' &= 0.
\end{aligned}$$

Moreover, the following bound holds.

$$\|(g' - e, k')\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} \lesssim \|(g - e, k)\|_{\mathcal{H}^w(B_1) \times \mathcal{H}^{w-1}(B_1)}.$$

4. THE DIVERGENCE EQUATION FOR k

In this section we prove the following theorem.

Theorem 4.1 (Extension of divergence-free tracefree symmetric 2-tensors, version 2). *There exists a small universal constant $\varepsilon > 0$ such that the following holds.*

- (1) Extension result: *Let $w \geq 2$ be an integer. Let g be a given $\mathcal{H}_{-1/2}^w$ -asymptotically flat metric on \mathbb{R}^3 and $\bar{k} \in \mathcal{H}^{w-1}(B_1)$ a symmetric 2-tensor such that on B_1*

$$\begin{aligned} \operatorname{div}_g \bar{k} &= 0, \\ \operatorname{tr}_g \bar{k} &= 0. \end{aligned} \tag{4.1}$$

If

$$\|g - e\|_{\mathcal{H}_{-1/2}^w} + \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)} < \varepsilon, \tag{4.2}$$

then there exists a symmetric 2-tensor $k \in \mathcal{H}_{-3/2}^{w-1}$ such that $k|_{B_1} = \bar{k}$ and on \mathbb{R}^3

$$\begin{aligned} \operatorname{div}_g k &= 0, \\ \operatorname{tr}_g k &= 0. \end{aligned}$$

Furthermore, it is bounded by

$$\|k\|_{\mathcal{H}_{-3/2}^{w-1}} \lesssim \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)}. \tag{4.3}$$

- (2) Iteration estimates: *Let g, g' be two given $\mathcal{H}_{-1/2}^w$ -asymptotically flat metrics on \mathbb{R}^3 such that*

$$g|_{B_1} = g'|_{B_1}$$

and $\bar{k} \in \mathcal{H}^{w-1}(B_1)$ a symmetric 2-tensor on B_1 that solves (4.1) with respect to g (and so for g'). Assume that for (g, \bar{k}) and (g', \bar{k}) the smallness condition (4.2) holds and let $k, k' \in \mathcal{H}_{-3/2}^{w-1}$ denote the two extensions of \bar{k} constructed in part (1) of this theorem with the metrics g and g' , respectively. Then it holds that

$$\|k - k'\|_{\mathcal{H}_{-3/2}^{w-1}} \lesssim \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)} \|g - g'\|_{\mathcal{H}_{-1/2}^w}. \tag{4.4}$$

Before proving Theorem 4.1, we analyse the divergence and trace mapping on $\mathcal{H}_{-1/2}^w$ -asymptotically flat metrics on \mathbb{R}^3 .

4.1. Analysis of operators on $\mathcal{H}_{-1/2}^w$ -asymptotically flat metrics. Recall from Section 2.1 that for a Riemannian metric g and a symmetric 2-tensor V , the divergence, trace and tracefree part of V is respectively defined as

$$\begin{aligned} (\operatorname{div}_g V)_j &:= \nabla^i V_{ij}, \\ \operatorname{tr}_g V &:= g^{ij} V_{ij}, \\ \widehat{V}^g &:= V - \frac{1}{3} \operatorname{tr}_g(V) g, \end{aligned}$$

where ∇ denotes the covariant derivative of g . The next lemma shows basic properties of the divergence operator.

Lemma 4.2. *Let $w \geq 2$ be an integer. There is a universal $\varepsilon > 0$ such that the following holds.*

- *The mapping*

$$\operatorname{div} : (V, g) \mapsto \operatorname{div}_g V$$

is a smooth mapping from $\mathcal{H}_{-3/2}^{w-1} \times B_\varepsilon(e)$ to $\mathcal{H}_{-5/2}^{w-2}$, where

$$B_\varepsilon(e) := \left\{ g : \|g - e\|_{\mathcal{H}_{-1/2}^w} < \varepsilon \right\}.$$

Furthermore, div maps $\overline{\mathcal{H}}_{-3/2}^{w-1} \times B_\varepsilon(e)$ into $\overline{\mathcal{H}}_{-5/2}^{w-2}$.

- *For all Riemannian metrics g such that*

$$\|g - e\|_{\mathcal{H}_{-1/2}^w} < \varepsilon,$$

it holds that

$$\|\operatorname{div}_g(V)\|_{\mathcal{H}_{-5/2}^{w-2}} \lesssim \|V\|_{\mathcal{H}_{-3/2}^{w-1}} \quad (4.5)$$

for all symmetric 2-tensors $V \in \mathcal{H}_{-3/2}^{w-1}$.

- *For all Riemannian metrics g, g' with*

$$\|g - e\|_{\mathcal{H}_{-1/2}^w}, \|g' - e\|_{\mathcal{H}_{-1/2}^w} < \varepsilon$$

it holds that

$$\|\operatorname{div}_g V - \operatorname{div}_{g'} V\|_{\mathcal{H}_{-5/2}^{w-2}} \lesssim \|g - g'\|_{\mathcal{H}_{-1/2}^w} \|V\|_{\mathcal{H}_{-3/2}^{w-1}} \quad (4.6)$$

for all symmetric 2-tensors $V \in \mathcal{H}_{-3/2}^{w-1}$.

Proof of Lemma 4.2. By definition of div_g ,

$$\begin{aligned} (\operatorname{div}_g V)_i &= g^{ab} \nabla_a V_{bi} \\ &= g^{ab} (\partial_a V_{bi} - \Gamma_{ab}^j V_{ji} - \Gamma_{ai}^j V_{jb}), \end{aligned} \quad (4.7)$$

where g^{ij} denotes the components of the inverse g^{-1} and $\Gamma_{ia}^j = \frac{1}{2} g^{jb} (\partial_i g_{ba} + \partial_a g_{bi} - \partial_b g_{ia})$ denote the Christoffel symbols. For g close to e , g^{ij} is a smooth expression in g . Therefore we can schematically write

$$\operatorname{div}_g V = F(g) \partial V + F(g) \partial g V, \quad (4.8)$$

where F maps symmetric 2-tensors into \mathbb{R} and is smooth in a neighbourhood of e . By (4.8),

$$\|\operatorname{div}_g V\|_{\mathcal{H}_{-5/2}^{w-2}} \lesssim \|F(g) \partial V\|_{\mathcal{H}_{-5/2}^{w-2}} + \|F(g) \partial g V\|_{\mathcal{H}_{-5/2}^{w-2}}. \quad (4.9)$$

By Lemmas 2.8 and 2.9, there exists a universal constant $\varepsilon > 0$ such that if $g \in B_\varepsilon(e) \subset \mathcal{H}_{-1/2}^w$, then

$$\begin{aligned} \|F(g)\partial V\|_{\mathcal{H}_{-5/2}^{w-2}} &\lesssim \|V\|_{\mathcal{H}_{-3/2}^{w-1}}, \\ \|F(g)\partial g V\|_{\mathcal{H}_{-5/2}^{w-2}} &\lesssim \|\partial g V\|_{\mathcal{H}_{-5/2}^{w-2}} \\ &\lesssim \|\partial g\|_{\mathcal{H}_{-3/2}^{w-1}} \|V\|_{\mathcal{H}_{-3/2}^{w-1}} \\ &\lesssim \|g - e\|_{\mathcal{H}_{-1/2}^w} \|V\|_{\mathcal{H}_{-3/2}^{w-1}}. \end{aligned} \tag{4.10}$$

Plugging (4.10) into (4.9) proves that div maps $\mathcal{H}_{-3/2}^{w-1} \times B_\varepsilon(e)$ to $\mathcal{H}_{-5/2}^{w-2}$ and further (4.5).

The expression (4.8) shows by Corollary 2.11 that div is a smooth mapping from $\mathcal{H}_{-3/2}^{w-1} \times B_\varepsilon(e)$ to $\mathcal{H}_{-5/2}^{w-2}$. The restriction of div to $V \in \overline{\mathcal{H}}_{-3/2}^{w-1}$ clearly maps into $\overline{\mathcal{H}}_{-5/2}^{w-2}$, by (4.8).

It remains to prove (4.6). Indeed, for all $g \in B_\varepsilon(e)$, $\text{div}_g(0) = 0$, so that Lemma 2.38 implies (4.6). This finishes the proof of Lemma 4.2. \square

Lemma 4.3. *Let $w \geq 2$ be an integer. There exists an $\varepsilon > 0$ such that the following holds.*

- *The mapping*

$$\text{tr} : (V, g) \mapsto \text{tr}_g V$$

is a smooth mapping from $\mathcal{H}_{-3/2}^{w-1} \times B_\varepsilon(e)$ to $\mathcal{H}_{-3/2}^{w-1}$, where

$$B_\varepsilon(e) := \left\{ g : \|g - e\|_{\mathcal{H}_{-1/2}^w} < \varepsilon \right\}.$$

Furthermore, tr maps $\overline{\mathcal{H}}_{-3/2}^{w-1} \times B_\varepsilon(e)$ into $\overline{\mathcal{H}}_{-5/2}^{w-2}$.

- *For all Riemannian metrics g with*

$$\|g - e\|_{\mathcal{H}_{-1/2}^w} < \varepsilon$$

it holds that

$$\|\text{tr}_g V\|_{\mathcal{H}_{-3/2}^{w-1}} \lesssim \|V\|_{\mathcal{H}_{-3/2}^{w-1}}.$$

for all symmetric 2-tensors $V \in \mathcal{H}_{-3/2}^{w-1}$.

- *For two metrics g, g' such that*

$$\|g - e\|_{\mathcal{H}_{-1/2}^w}, \|g' - e\|_{\mathcal{H}_{-1/2}^w} < \varepsilon$$

it holds that

$$\|\text{tr}_g V - \text{tr}_{g'} V\|_{\mathcal{H}_{-3/2}^{w-1}} \lesssim \|g - g'\|_{\mathcal{H}_{-1/2}^w} \|V\|_{\mathcal{H}_{-3/2}^{w-1}}$$

for all symmetric 2-tensors $V \in \mathcal{H}_{-3/2}^{w-1}$.

The proof of Lemma 4.3 is similar to the proof of Lemma 4.2 and left to the reader.

Lemmas 4.2 and 4.3 imply the following corollary. The proof is left to the reader.

Corollary 4.4. *Let $w \geq 2$ be an integer. There exists $\varepsilon > 0$ such that the following holds.*

- *The mapping*

$$(V, g) \mapsto \operatorname{div}_g (\widehat{V}^g)$$

is smooth from $\mathcal{H}_{-3/2}^{w-1} \times B_\varepsilon(e)$ to $\mathcal{H}_{-5/2}^{w-2}$, where

$$B_\varepsilon(e) := \left\{ g : \|g - e\|_{\mathcal{H}_{-1/2}^w} < \varepsilon \right\}.$$

Furthermore, the restriction of this mapping to $V \in \overline{\mathcal{H}}_{-3/2}^{w-1}$ maps into $\overline{\mathcal{H}}_{-5/2}^{w-2}$.

- *For a Riemannian metric g on \mathbb{R}^3 such that*

$$\|g - e\|_{\mathcal{H}_{-1/2}^w} < \varepsilon, \tag{4.11}$$

it holds that

$$\begin{aligned} \|\widehat{V}^g\|_{\mathcal{H}_{-3/2}^{w-1}} &\lesssim \|V\|_{\mathcal{H}_{-3/2}^{w-1}}, \\ \left\| \operatorname{div}_g (\widehat{V}^g) \right\|_{\mathcal{H}_{-5/2}^{w-2}} &\lesssim \|V\|_{\mathcal{H}_{-3/2}^{w-1}}. \end{aligned}$$

for all symmetric 2-tensors $V \in \mathcal{H}_{-3/2}^{w-1}$.

- *For two Riemannian metrics g, g' on \mathbb{R}^3 such that*

$$\|g - e\|_{\mathcal{H}_{-1/2}^w}, \|g' - e\|_{\mathcal{H}_{-1/2}^w} < \varepsilon,$$

it holds that

$$\begin{aligned} \left\| \widehat{V}^g - \widehat{V}^{g'} \right\|_{\mathcal{H}_{-3/2}^{w-1}} &\lesssim \|g - g'\|_{\mathcal{H}_{-1/2}^w} \|V\|_{\mathcal{H}_{-3/2}^{w-1}}, \\ \left\| \operatorname{div}_g (\widehat{V}^g) - \operatorname{div}_{g'} (\widehat{V}^{g'}) \right\|_{\mathcal{H}_{-5/2}^{w-2}} &\lesssim \|g - g'\|_{\mathcal{H}_{-1/2}^w} \|V\|_{\mathcal{H}_{-3/2}^{w-1}} \end{aligned}$$

for all symmetric 2-tensors $V \in \mathcal{H}_{-3/2}^{w-1}$.

4.2. Reduction to the Euclidean case. In this section, we prove Theorem 4.1 under the assumption of Lemma 4.6 below which is proved in Section 4.3. First, as an intermediate step, we prove the next proposition.

Proposition 4.5. *Let $w \geq 2$ be an integer. There is a universal constant $\varepsilon > 0$ such that the following holds.*

- (1) *Let g be an $\mathcal{H}_{-1/2}^w$ -asymptotically flat metric and $\rho \in \overline{\mathcal{H}}_{-5/2}^{w-2}$ a 1-form on \mathbb{R}^3 such that*

$$\|g - e\|_{\mathcal{H}_{-1/2}^w} + \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}} < \varepsilon. \tag{4.12}$$

Then there exists $k \in \overline{\mathcal{H}}_{-3/2}^{w-1}$ solving on $\mathbb{R}^3 \setminus \overline{B_1}$

$$\begin{aligned}\operatorname{div}_g k &= \rho, \\ \operatorname{tr}_g k &= 0\end{aligned}$$

and bounded by

$$\|k\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} \lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}}. \quad (4.13)$$

(2) Moreover, for two pairs $(g, \rho), (g', \rho')$ satisfying the smallness condition (4.12), the respectively constructed k, k' satisfy

$$\|k - k'\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} \lesssim \|\rho - \rho'\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}} + \|g - g'\|_{\mathcal{H}_{-1/2}^w} \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}}. \quad (4.14)$$

To prove Proposition 4.5, we assume the following essential lemma proved in Section 4.3.

Lemma 4.6 (Surjectivity at the Euclidean metric). *Let $w \geq 2$ be an integer. For any $\rho \in \overline{\mathcal{H}}_{-5/2}^{w-2}$, there exists a symmetric 2-tensor $k \in \overline{\mathcal{H}}_{-3/2}^{w-1}$ solving on $\mathbb{R}^3 \setminus \overline{B_1}$*

$$\begin{aligned}\operatorname{div}_e k &= \rho, \\ \operatorname{tr}_e k &= 0\end{aligned}$$

and bounded by

$$\|k\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} \lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}}.$$

In other words, the mapping $k \mapsto \operatorname{div}_e(\hat{k}^e)$ from $\overline{\mathcal{H}}_{-3/2}^{w-1}$ to $\overline{\mathcal{H}}_{-5/2}^{w-2}$ is surjective and has a bounded right-inverse.

For the rest of this section denote

$$\overline{\mathcal{N}}_e := \ker(\operatorname{div}_e \circ (\hat{\cdot}^e))^\perp, \quad (4.15)$$

where $^\perp$ denotes the orthogonal complement with respect to the scalar product on $\overline{\mathcal{H}}_{-3/2}^{w-1}$. $\overline{\mathcal{N}}_e$ is a closed subspace of the Hilbert space $\overline{\mathcal{H}}_{-3/2}^{w-1}$ and therefore Hilbert itself.

We are now ready to prove Proposition 4.5 by Lemma 4.6 and the Implicit Function Theorem 2.37.

Proof of Proposition 4.5. We prove each part separately.

Proof of part (1). We apply the Implicit Function Theorem 2.37 to the mapping

$$\begin{aligned}\mathcal{F} : \overline{\mathcal{N}}_e \times \mathcal{H}_{-1/2}^w &\rightarrow \overline{\mathcal{H}}_{-5/2}^{w-2} \\ (k, h) &\mapsto \rho := \operatorname{div}_{e+h}(\hat{k}^{e+h}),\end{aligned}$$

where h is a symmetric 2-tensor. We verify that \mathcal{F} satisfies the assumptions of Theorem 2.37 at $(k, h) = 0$. On the one hand, by Corollary 4.4, there exists an $\varepsilon > 0$ such that \mathcal{F}

is a smooth mapping from $\overline{\mathcal{N}}_e \times B_\varepsilon(0)$ to $\overline{\mathcal{H}}_{-5/2}^{w-2}$, where $B_\varepsilon(0) \subset \mathcal{H}_{-1/2}^w$, and $\mathcal{F}(0,0) = 0$. On the other hand, by Lemma 4.6 and the definition of $\overline{\mathcal{N}}_e$ in (4.15), the linearization in the first argument at $h = 0$,

$$D_1 \mathcal{F}|_{h=0} : \overline{\mathcal{N}}_e \rightarrow \overline{\mathcal{H}}_{-5/2}^{w-2},$$

is an isomorphism.

Consequently, by Theorem 2.37, there exists an open neighbourhood $V_0 \subset B_\varepsilon(0) \times \overline{\mathcal{H}}_{-5/2}^{w-2}$ of $(h, \rho) = (0, 0)$ and a unique smooth mapping $\mathcal{G} : V_0 \rightarrow \overline{\mathcal{H}}_{-3/2}^{w-1}$ into symmetric 2-tensors such that on $\mathbb{R}^3 \setminus \overline{B_1}$

$$\operatorname{div}_{e+h} \left(\widehat{\mathcal{G}}^{e+h}(h, \rho) \right) = \rho$$

for all $(h, \rho) \in V_0$. By the uniqueness it follows in particular that for all $(h, 0) \in V_0$,

$$\mathcal{G}(h, 0) = 0, \tag{4.16}$$

because $\mathcal{F}(0, h) = 0$ for all $h \in B_\varepsilon(0)$.

For $(h, \rho) \in V_0$, let $k := \widehat{\mathcal{G}}^{e+h}(h, \rho)$. Then, on $\mathbb{R}^3 \setminus \overline{B_1}$,

$$\begin{aligned} \operatorname{div}_{e+h} k &= \rho, \\ \operatorname{tr}_{e+h} k &= 0. \end{aligned}$$

For $\varepsilon > 0$ sufficiently small it holds that

$$(h, \rho) \in B_\varepsilon(0) \times B_\varepsilon(0) \subset V_0$$

and further by Lemma 2.38 and Corollary 4.4,

$$\begin{aligned} \|k\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} &= \|\widehat{\mathcal{G}}^{e+h}(h, \rho)\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} \\ &\lesssim \|\mathcal{G}(h, \rho)\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} \\ &\lesssim \|\mathcal{G}(h, \rho) - \underbrace{\mathcal{G}(h, 0)}_{=0}\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} \\ &\lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}}. \end{aligned}$$

This proves (4.13).

Proof of part (2). Let two pairs $(h, \rho), (h', \rho') \in B_\varepsilon(0) \times B_\varepsilon(0) \subset V_0$. By Lemma 2.38 and (4.16), it follows that for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned} \|\mathcal{G}(h, \rho) - \mathcal{G}(h', \rho')\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} &\lesssim \|h - h'\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3)} \|\rho\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}}, \\ \|\mathcal{G}(h, \rho)\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} &\lesssim \|\rho\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}}. \end{aligned} \tag{4.17}$$

Moreover, by the smoothness of \mathcal{G} , for $\varepsilon > 0$ sufficiently small,

$$\|\mathcal{G}(h', \rho) - \mathcal{G}(h', \rho')\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} \lesssim \|\rho - \rho'\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}}. \quad (4.18)$$

Let $k := \widehat{\mathcal{G}}^{e+h}(h, \rho)$, $k' := \widehat{\mathcal{G}}^{e+h'}(h', \rho')$. By (4.17) and (4.18), Lemma 4.3 and Corollary 4.4, for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned} \|k - k'\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} &\lesssim \|[\mathcal{G}(h, \rho) - \mathcal{G}(h', \rho')]^{\wedge_{e+h'}}\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} + \|\mathrm{tr}_{e+h'} \mathcal{G}(h, \rho) - \mathrm{tr}_{e+h} \mathcal{G}(h, \rho)\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} \\ &\lesssim \|\mathcal{G}(h, \rho) - \mathcal{G}(h', \rho')\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} + \|h' - h\|_{\mathcal{H}_{-1/2}^w} \|\mathcal{G}(h, \rho)\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} \\ &\lesssim \|\mathcal{G}(h, \rho) - \mathcal{G}(h', \rho)\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} + \|\mathcal{G}(h', \rho) - \mathcal{G}(h', \rho')\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} \\ &\quad + \|h' - h\|_{\mathcal{H}_{-1/2}^w} \|\mathcal{G}(h, \rho)\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} \\ &\lesssim \|h' - h\|_{\mathcal{H}_{-1/2}^w} \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}} + \|\rho - \rho'\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}}. \end{aligned}$$

This finishes the proof of Proposition 4.5. \square

We now turn to the proof of Theorem 4.1.

Proof of Theorem 4.1. We prove the two parts of Theorem 4.1 separately.

Proof of Part 1: Let the symmetric 2-tensor $\bar{k} \in \mathcal{H}^{w-1}(B_1)$ solve on B_1

$$\begin{aligned} \mathrm{div}_g \bar{k} &= 0, \\ \mathrm{tr}_g \bar{k} &= 0. \end{aligned}$$

Using standard Sobolev extension, see for example Theorem 7.25 in [16], continuously extend \bar{k} to a symmetric 2-tensor $\check{k} \in \mathcal{H}_{loc}^{w-1}(\mathbb{R}^3)$. We can assume without loss of generality that \check{k} is g -tracefree and

$$\|\check{k}\|_{\mathcal{H}_{-3/2}^{w-1}} \lesssim \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)}. \quad (4.19)$$

Indeed, for $\|g - e\|_{\mathcal{H}_{-1/2}^w}$ small enough, multiplying by a cut-off function and taking the g -tracefree part are both continuous endomorphisms of $\mathcal{H}_{loc}^{w-1}(\mathbb{R}^3)$, see Corollary 4.4.

Let $\check{\rho} := \mathrm{div}_g \check{k}$. For $\varepsilon > 0$ small enough, by Lemma 4.2 and (4.19),

$$\begin{aligned} \|\check{\rho}\|_{\mathcal{H}_{-5/2}^{w-2}} &\lesssim \|\check{k}\|_{\mathcal{H}_{-3/2}^{w-1}} \\ &\lesssim \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)}. \end{aligned} \quad (4.20)$$

Further, it holds that on B_1

$$\check{\rho} = \mathrm{div}_g \check{k} = \mathrm{div}_g \bar{k} = 0,$$

so by Proposition 2.13, $\check{\rho} \in \overline{\mathcal{H}}_{-5/2}^{w-2}$. This $\check{\rho}$ is in general non-trivial (otherwise we would be done), and the Sobolev extension \check{k} is therefore in general not a solution to (4.1).

We have by (4.20)

$$\|g - e\|_{\mathcal{H}_{-1/2}^w} + \|\check{\rho}\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} \lesssim \|g - e\|_{\mathcal{H}_{-1/2}^w} + \|\bar{k}\|_{H^{w-1}(B_1)}.$$

Therefore, for $\varepsilon > 0$ small enough, Proposition 4.5 yields a symmetric 2-tensor $\tilde{k} \in \overline{\mathcal{H}}_{-3/2}^{w-1}$ that solves on $\mathbb{R}^3 \setminus \overline{B_1}$

$$\begin{aligned} \operatorname{div}_g \tilde{k} &= -\check{\rho}, \\ \operatorname{tr}_g \tilde{k} &= 0 \end{aligned}$$

and is bounded by

$$\|\tilde{k}\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} \lesssim \|\check{\rho}\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}}. \quad (4.21)$$

Extend \tilde{k} trivially to B_1 . By Proposition 2.13, $\tilde{k} \in \mathcal{H}_{-3/2}^{w-1}$. Consequently, the symmetric 2-tensor

$$k := \check{k} + \tilde{k} \in \mathcal{H}_{-3/2}^{w-1} \quad (4.22)$$

is such that $k|_{B_1} = \bar{k}$ and solves on \mathbb{R}^3

$$\begin{aligned} \operatorname{div}_g k &= 0, \\ \operatorname{tr}_g k &= 0. \end{aligned}$$

Finally, for $\varepsilon > 0$ sufficiently small, by the estimates (4.20) and (4.21),

$$\|k\|_{\mathcal{H}_{-3/2}^{w-1}} \lesssim \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)}.$$

This proves the first part of Theorem 4.1.

Proof of Part 2: Extend $\bar{k} \in \mathcal{H}^{w-1}(B_1)$ by standard Sobolev extension to a symmetric 2-tensor $\check{k} \in \mathcal{H}_{-3/2}^{w-1}$ on \mathbb{R}^3 such that

$$\|\check{k}\|_{\mathcal{H}_{-3/2}^{w-1}} \lesssim \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)}. \quad (4.23)$$

Taking the g -tracefree and g' -tracefree parts of \check{k} yields two symmetric 2-tensors $\widehat{\check{k}}^g$ and $\widehat{\check{k}}^{g'} \in \mathcal{H}_{-3/2}^{w-1}$, respectively, that both extend \bar{k} and satisfy for $\varepsilon > 0$ sufficiently small

$$\begin{aligned} \left\| \widehat{\check{k}}^g \right\|_{\mathcal{H}_{-3/2}^{w-1}} &\lesssim \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)}, \\ \left\| \widehat{\check{k}}^{g'} \right\|_{\mathcal{H}_{-3/2}^{w-1}} &\lesssim \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)}. \end{aligned}$$

By Proposition 2.13,

$$\rho := \operatorname{div}_g \widehat{\check{k}}^g \in \overline{\mathcal{H}}_{-5/2}^{w-2}, \rho' := \operatorname{div}_{g'} \widehat{\check{k}}^{g'} \in \overline{\mathcal{H}}_{-5/2}^{w-2}.$$

For $\varepsilon > 0$ sufficiently small, by Lemma 4.2 and (4.23),

$$\|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}}, \|\rho'\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}} \lesssim \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)}. \quad (4.24)$$

For $\varepsilon > 0$ small enough, applying Proposition 4.5 to ρ, ρ' with metrics g, g' yields two tensors $\tilde{k}, \tilde{k}' \in \overline{\mathcal{H}}_{-3/2}^{w-1}$, respectively, that satisfy

$$\begin{aligned} \operatorname{div}_g \tilde{k} &= -\rho, \\ \operatorname{tr}_g \tilde{k} &= 0, \\ \operatorname{div}_{g'} \tilde{k}' &= -\rho', \\ \operatorname{tr}_{g'} \tilde{k}' &= 0. \end{aligned}$$

By (4.14) in Proposition 4.5, for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned} \|\tilde{k} - \tilde{k}'\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} &\lesssim \|g - g'\|_{\mathcal{H}_{-1/2}^w} \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}} + \|\rho - \rho'\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}} \\ &\lesssim \|g - g'\|_{\mathcal{H}_{-1/2}^w} \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)} + \left\| \operatorname{div}_g \widehat{\tilde{k}}^g - \operatorname{div}_{g'} \widehat{\tilde{k}}^{g'} \right\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}} \\ &\lesssim \|g - g'\|_{\mathcal{H}_{-1/2}^w} \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)} + \|g - g'\|_{\mathcal{H}_{-1/2}^w} \|\tilde{k}\|_{\mathcal{H}_{-3/2}^{w-1}} \\ &\lesssim \|g - g'\|_{\mathcal{H}_{-1/2}^w} \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)}, \end{aligned} \quad (4.25)$$

where we used (4.24) and Corollary 4.4. Extend \tilde{k}, \tilde{k}' trivially to B_1 . By Proposition 2.13, $\tilde{k}, \tilde{k}' \in \mathcal{H}_{-3/2}^{w-1}$.

The tensors

$$k := \widehat{\tilde{k}}^g + \tilde{k} \in \mathcal{H}_{-3/2}^{w-1}, k' := \widehat{\tilde{k}}^{g'} + \tilde{k}' \in \mathcal{H}_{-3/2}^{w-1}$$

both extend \bar{k} and satisfy on \mathbb{R}^3

$$\begin{aligned} \operatorname{div}_g k &= 0, \\ \operatorname{tr}_g k &= 0, \\ \operatorname{div}_{g'} k' &= 0, \\ \operatorname{tr}_{g'} k' &= 0. \end{aligned}$$

Moreover, their difference is bounded by

$$\begin{aligned} \|k - k'\|_{\mathcal{H}_{-3/2}^{w-1}} &\leq \left\| \widehat{\tilde{k}}^g - \widehat{\tilde{k}}^{g'} \right\|_{\mathcal{H}_{-3/2}^{w-1}} + \|\tilde{k} - \tilde{k}'\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} \\ &\lesssim \|g - g'\|_{\mathcal{H}_{-1/2}^w} \|\tilde{k}\|_{\mathcal{H}_{-3/2}^{w-1}} + \|g - g'\|_{\mathcal{H}_{-1/2}^w} \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)} \\ &\lesssim \|g - g'\|_{\mathcal{H}_{-1/2}^w} \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)}, \end{aligned}$$

where we used Corollary 4.4, (4.23) and (4.25). This finishes the proof of Theorem 4.1. \square

4.3. Surjectivity at the Euclidean metric. Let $w \geq 2$ be an integer. In this section we prove Lemma 4.6, that is, we show that for any $\rho \in \overline{\mathcal{H}}_{-5/2}^{w-2}$, there exists a symmetric 2-tensor $k \in \overline{\mathcal{H}}_{-3/2}^{w-1}$ that solves on $\mathbb{R}^3 \setminus \overline{B_1}$

$$\begin{aligned} \operatorname{div}_e k &= \rho, \\ \operatorname{tr}_e k &= 0 \end{aligned}$$

and is bounded by

$$\|k\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} \lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}}.$$

In this section, all differential operators are with respect to the Euclidean metric e . The operators div , curl , ∇ are the induced operators on the spheres $(S_r, \overset{\circ}{\gamma}) \subset (\mathbb{R}^3, e)$ for $r > 0$.

Remark 4.7. *Let us note the following.*

- In general, the system on $\mathbb{R}^3 \setminus \overline{B_1}$

$$\begin{aligned} \operatorname{div} k &= \rho, \\ \operatorname{tr} k &= 0 \end{aligned} \tag{4.26}$$

is underdetermined and does not admit an a priori estimate for solutions k . We work with the following determined Hodge system on $\mathbb{R}^3 \setminus \overline{B_1}$

$$\begin{aligned} \operatorname{div} k &= \rho, \\ \operatorname{curl} k &= \sigma, \\ \operatorname{tr} k &= 0, \end{aligned} \tag{4.27}$$

where σ is a tracefree symmetric 2-tensor that we carefully choose by hand. This system admits in general a priori estimates for k in terms of ρ and σ , see for example Proposition 4.4.1 in [9]. Clearly, a solution k to (4.27) is in particular a solution to (4.26).

- First, we decompose k with respect to the foliation of $\mathbb{R}^3 \setminus \{0\}$ by spheres S_r into scalar functions and S_r -tangent tensors. Second, the Hodge-Fourier expansion of S_r -tangent tensors introduced in Section 2 allows to decompose (4.27) into three independent sub-systems **S0**, **S1** and **S2**, see later in this section. These sub-systems are then solved individually.
- A priori, for given $\rho, \sigma \in \overline{\mathcal{H}}_{-5/2}^{w-2}$, the solution k to the Dirichlet boundary value problem on $\mathbb{R}^3 \setminus \overline{B_1}$

$$\begin{cases} \operatorname{div} k = \rho, \\ \operatorname{curl} k = \sigma, \\ \operatorname{tr} k = 0, \\ k|_{r=1} = 0 \end{cases}$$

satisfies only $k \in \mathcal{H}_{-3/2}^{w-1}(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{\mathcal{H}}_{-3/2}^1$. However, by our careful choice of σ we can achieve the additional boundary condition

$$\nabla_N k|_{r=1} = 0,$$

which eventually improves the boundary behaviour to $k \in \overline{\mathcal{H}}_{-3/2}^{w-1}$.

4.3.1. *Derivation of the equations.* In this section, we derive the new form of (4.27) with respect to the radial foliation of $\mathbb{R}^3 \setminus \{0\}$, see Section 2.2 for notations. Decompose the tensor k into

- the scalar $\delta := k_{NN}$,
- the S_r -tangent vectorfield $\epsilon_A := (k_N)_A$,
- the S_r -tangent symmetric 2-tensor $\eta_{AB} := (k)_{AB}$.

Furthermore, let $\hat{\eta}$ be the tracefree part of η , that is⁵

$$\hat{\eta}_{AB} := \eta_{AB} + \frac{1}{2}\delta\gamma_{AB}.$$

Decompose ρ into

- the scalar ρ_N ,
- the S_r -tangent vectorfield $\not{\rho}_A := \rho_A$,

and σ into

- the scalar σ_{NN} ,
- the S_r -tangent vectorfield $\sigma_{\not{N}A} := \sigma_{AN}$,
- the S_r -tangent symmetric 2-tensor $\not{\phi}_{AB} := \sigma_{AB}$.

The system (4.27) is equivalent to (this is the Euclidean version of Proposition 4.4.3 in [9])

$$\begin{aligned} \operatorname{div} \epsilon &= \rho_N - \nabla_N \delta - \frac{3}{r}\delta, \\ \operatorname{curl} \epsilon &= \sigma_{NN}, \\ \nabla_N \epsilon + \frac{2}{r}\epsilon &= \frac{1}{2}\not{\rho} + {}^*\sigma_{\not{N}} + \nabla \delta, \\ \operatorname{div} \hat{\eta} &= \frac{1}{2}\not{\rho} - {}^*\sigma_{\not{N}} - \frac{1}{2}\nabla \delta - \frac{1}{r}\epsilon, \\ \nabla_N \hat{\eta} + \frac{1}{r}\hat{\eta} &= {}^*(\widehat{\not{\phi}}) + \frac{1}{2}\nabla \widehat{\otimes} \epsilon, \end{aligned} \tag{4.28}$$

where ${}^*\sigma_{\not{N}}$ denotes the Hodge dual of $\sigma_{\not{N}}$ and ${}^*(\widehat{\not{\phi}})$ the Hodge dual of $\widehat{\not{\phi}}$, the tracefree part of $\not{\phi}$. See Section 2.6 for details.

⁵Here we use that $\gamma^{\circ AB} k_{AB} = -\delta$ by the third equation of (4.27).

4.3.2. *Definition of the 2-tensors k and σ .* In this section, we explicitly exhibit the two symmetric 2-tensors k and σ . We show in Section 4.3.3 that they form a regular solution to (4.28).

Let $\rho = (\rho_N, \not\rho) \in \overline{\mathcal{H}}_{-3/2}^{w-1}$. Let the Hodge-Fourier decomposition of $\rho_N, \not\rho$ be

$$\begin{aligned}\rho_N &= (\rho_N)^{[0]} + (\rho_N)^{[1]} + (\rho_N)^{[\geq 2]}, \\ \not\rho &= \not\rho_E^{[0]} + \not\rho_H^{[0]} + \not\rho_E^{[1]} + \not\rho_H^{[1]} + \not\rho_E^{[\geq 2]} + \not\rho_H^{[\geq 2]}.\end{aligned}$$

Define symmetric tracefree 2-tensors k and σ on $\mathbb{R}^3 \setminus \overline{B_1}$ as follows.

- **Definition of δ .** Let the scalar function

$$\delta = \delta^{[0]} + \delta^{[1]} + \delta^{[\geq 2]}, \quad (4.29)$$

where $\delta^{[0]}$ is defined as

$$\delta^{[0]} := \frac{1}{r^3} \int_1^r (r')^3 (\rho_N)^{[0]} dr' \quad (4.30)$$

and $\delta^{[1]}$ is defined as the solution to the second-order ODE on $r > 1$

$$\begin{cases} \partial_r^2 \delta^{[1]} + \frac{7}{r} \partial_r \delta^{[1]} + \frac{8}{r^2} \delta^{[1]} = \frac{1}{r^4} \partial_r (r^4 (\rho_N)^{[1]}) - \text{div} \not\rho^{[1]}, \\ \delta^{[1]}|_{r=1} = \partial_r \delta^{[1]}|_{r=1} = 0. \end{cases} \quad (4.31)$$

The function $\delta^{[\geq 2]}$ is defined as the solution to the following elliptic boundary value problem on $\mathbb{R}^3 \setminus \overline{B_1}$,

$$\begin{cases} \Delta \delta^{[\geq 2]} + \frac{4}{r} \partial_r \delta^{[\geq 2]} + \frac{6}{r^2} \delta^{[\geq 2]} = \frac{1}{r^3} \partial_r \left(r^3 (\rho_N)^{[\geq 2]} \right) - \text{div} \left(\not\rho_E^{[\geq 2]} + \zeta_E \right), \\ \delta^{[\geq 2]}|_{r=1} = 0. \end{cases} \quad (4.32)$$

Here, the S_r -tangent vectorfield ζ_E is defined on \mathbb{R}^3 by

$$\begin{aligned}\zeta_E &:= \sum_{l \geq 2} \sum_{m=-l}^l \zeta_E^{(lm)} E^{(lm)}, \\ \zeta_E^{(lm)}(r) &:= c_E^{(lm)} r^{l-1} \partial_r (\chi(l(r-1))),\end{aligned} \quad (4.33)$$

where χ is the standard transition function defined in (2.1) and for $l \geq 2$,

$$c_E^{(lm)} := \int_1^\infty r^{-l+1} \left(\frac{l}{\sqrt{l(l+1)}} (\rho_N)^{(lm)} - \not\rho_E^{(lm)} \right) dr. \quad (4.34)$$

- **Definition of σ_{NN} .** Let the scalar function

$$\sigma_{NN} = \sigma_{NN}^{[1]} + \sigma_{NN}^{[\geq 2]}, \quad (4.35)$$

where $\sigma_{NN}^{[1]}$ is defined as

$$\sigma_{NN}^{[1]} := \frac{1}{r^4} \int_1^r (r')^4 \operatorname{curl} \rho^{[1]} dr', \quad (4.36)$$

and $\sigma_{NN}^{[\geq 2]}$ is defined as solution to the following elliptic boundary value problem on $\mathbb{R}^3 \setminus \overline{B_1}$,

$$\begin{cases} \Delta \sigma_{NN}^{[\geq 2]} + \frac{1}{r} \partial_r \sigma_{NN}^{[\geq 2]} - \frac{3}{r^2} \sigma_{NN}^{[\geq 2]} = \partial_r \left(\operatorname{curl} \left(\rho_H^{[\geq 2]} + \zeta_H^{[\geq 2]} \right) \right), \\ \sigma_{NN}^{[\geq 2]}|_{r=1} = 0. \end{cases} \quad (4.37)$$

Here, the S_r -tangent vectorfield ζ_H is defined by

$$\begin{aligned} \zeta_H &:= \sum_{l \geq 2} \sum_{m=-l}^l \zeta_H^{(lm)} H^{(lm)}, \\ \zeta_H^{(lm)}(r) &:= c_H^{(lm)} r^{1+\sqrt{l(l+1)+4}} \partial_r (\chi(l(r-1))), \end{aligned} \quad (4.38)$$

and for $l \geq 2$,

$$c_H^{(lm)} := - \int_1^\infty r^{-1-\sqrt{l(l+1)+4}} \rho_H^{(lm)} dr. \quad (4.39)$$

- **Definition of ϵ .** Let the S_r -tangent vectorfield ϵ be on each S_r , $r \geq 1$, the solution to

$$\mathcal{P}_1 \epsilon = \left(\rho_N - \frac{1}{r^3} \partial_r (r^3 \delta), \sigma_{NN} \right). \quad (4.40)$$

- **Definition of ${}^* \sigma_N$.** Let the S_r -tangent vectorfield

$${}^* \sigma_N = {}^* \sigma_N^{[1]} + {}^* \sigma_{NE}^{[\geq 2]} + {}^* \sigma_{NH}^{[\geq 2]}, \quad (4.41)$$

where ${}^* \sigma_N^{[1]}$, ${}^* \sigma_{NE}^{[\geq 2]}$ are defined as

$${}^* \sigma_N^{[1]} := \frac{1}{2} \rho^{[1]} - \frac{1}{2} \nabla \delta^{[1]} - \frac{1}{r} \epsilon^{[1]}, \quad (4.42)$$

$${}^* \sigma_{NE}^{[\geq 2]} := \frac{1}{2} \rho_E^{[\geq 2]} + \zeta_E, \quad (4.43)$$

and ${}^* \sigma_{NH}^{[\geq 2]}$ is defined to be on each S_r , $r \geq 1$, the solution of

$$\mathcal{P}_1 \left({}^* \sigma_{NH}^{[\geq 2]} \right) = -\mathcal{P}_1 \left(\frac{1}{2} \rho_H^{[\geq 2]} \right) + \left(0, \frac{1}{r^3} \partial_r (r^3 \sigma_{NN}^{[\geq 2]}) \right). \quad (4.44)$$

- **Construction of $^*(\widehat{\phi})$.** Let the symmetric $\overset{\circ}{\gamma}$ -tracefree 2-tensor $^*(\widehat{\phi})$ be on each S_r , $r \geq 1$, the solution to

$$\begin{aligned} \mathcal{D}_2 \left(^*(\widehat{\phi}) \right) = & - \mathcal{D}_2 \left(\frac{1}{2} \nabla \widehat{\epsilon}^{[\geq 2]} \right) \\ & + \frac{1}{r^2} \nabla_N \left(r^2 \left(\frac{1}{2} \rho^{[\geq 2]} - ^*\sigma_N^{[\geq 2]} - \frac{1}{2} \nabla \delta^{[\geq 2]} - \frac{1}{r} \epsilon^{[\geq 2]} \right) \right). \end{aligned} \quad (4.45)$$

- **Construction of $\hat{\eta}$.** Let the symmetric $\overset{\circ}{\gamma}$ -tracefree 2-tensor $\hat{\eta}$ be on each S_r , $r \geq 1$, the solution to

$$\mathcal{D}_2 \hat{\eta} = \frac{1}{2} \rho^{[\geq 2]} - ^*\sigma_N^{[\geq 2]} - \frac{1}{2} \nabla \delta^{[\geq 2]} - \frac{1}{r} \epsilon^{[\geq 2]}. \quad (4.46)$$

Remark 4.8. For ease of presentation, we defined k and σ via the quantities that appear in (4.28). Indeed, by the Hodge duality relation (2.8) and the third equation of (4.27), all components of k and σ are uniquely specified this way.

Remark 4.9. The auxiliary ζ_E and ζ_H in (4.38) and (4.38) are introduced to control the Dirichlet-to-Neumann map of the elliptic boundary value problems (4.32) and (4.37) for $\delta^{[\geq 2]}$ and $\sigma_{NN}^{[\geq 2]}$, respectively, to achieve the additional boundary condition

$$\partial_r \delta^{[\geq 2]}|_{r=1} = \partial_r \sigma_{NN}^{[\geq 2]}|_{r=1} = 0.$$

This is necessary for the higher boundary regularity of $\delta^{[\geq 2]}$ and $\sigma_{NN}^{[\geq 2]}$ across S_1 .

4.3.3. Proof of surjectivity. In this section, we prove Lemma 4.10, Proposition 4.11 and Lemma 4.15 below, that together imply surjectivity. Especially Proposition 4.11 is essential and only holds due to our delicate choice of ζ_E, ζ_H in (4.33) and (4.38), as well as our particular choice of $\sigma_{NN}^{[\geq 2]}$ to be a solution to (4.37). See also Remark 4.9.

Lemma 4.10. For given ρ , the symmetric 2-tensors k and σ defined by (4.29)-(4.46) are a formal solution to (4.28), that is, on $\mathbb{R}^3 \setminus \overline{B_1}$,

$$\begin{aligned} \operatorname{div} k &= \rho, \\ \operatorname{curl} k &= \sigma, \\ \operatorname{tr} k &= 0. \end{aligned}$$

Proof. The Hodge system (4.28) is linear and its coefficients depend only on r . Therefore, we may project the equations of (4.28) onto the Hodge-Fourier basis elements. This uses Remark 2.26 and Proposition 2.30. We split (4.28) into the modes $l = 0, 1$ and $l \geq 2$, which yields the following three subsystems **S0**, **S1** and **S2**.

$$0 = (\rho_N)^{[0]} - \frac{1}{r^3} \partial_r (r^3 \delta^{[0]}), \quad (\text{S0.1})$$

$$0 = \sigma_{NN}^{[0]}, \quad (\text{S0.2})$$

$$\mathrm{d}\!\!\!\diagup\epsilon^{[1]} = (\rho_N)^{[1]} - \frac{1}{r^3}\partial_r (r^3\delta^{[1]}), \quad (\text{S1.1})$$

$$\mathrm{cu}\!\!\!\diagup\epsilon^{[1]} = \sigma_{NN}^{[1]}, \quad (\text{S1.2})$$

$$\frac{1}{r^3}\nabla_N (r^3\epsilon^{[1]}) = \not\partial^{[1]} + \frac{1}{2}\nabla\delta^{[1]}, \quad (\text{S1.3})$$

$$*\sigma_N^{[1]} = \frac{1}{2}\not\partial^{[1]} - \frac{1}{2}\nabla\delta^{[1]} - \frac{1}{r}\epsilon^{[1]}, \quad (\text{S1.4})$$

and, using that $\hat{\eta}^{[\geq 2]} = \hat{\eta}, *\widehat{(\not\partial)}^{[\geq 2]} = *\widehat{(\not\partial)}$,

$$\mathrm{d}\!\!\!\diagup\epsilon^{[\geq 2]} = \rho_N^{[\geq 2]} - \frac{1}{r^3}\partial_r (r^3\delta^{[\geq 2]}), \quad (\text{S2.1})$$

$$\mathrm{cu}\!\!\!\diagup\epsilon^{[\geq 2]} = \sigma_{NN}^{[\geq 2]}, \quad (\text{S2.2})$$

$$\frac{1}{r^2}\nabla_N (r^2\epsilon^{[\geq 2]}) = \frac{1}{2}\not\partial^{[\geq 2]} + *\sigma_N^{[\geq 2]} + \nabla\delta^{[\geq 2]}, \quad (\text{S2.3})$$

$$\mathrm{d}\!\!\!\diagup\hat{\eta} = \frac{1}{2}\not\partial^{[\geq 2]} - *\sigma_N^{[\geq 2]} - \frac{1}{2}\nabla\delta^{[\geq 2]} - \frac{1}{r}\epsilon^{[\geq 2]}, \quad (\text{S2.4})$$

$$\nabla_N\hat{\eta} + \frac{1}{r}\hat{\eta} = *\widehat{(\not\partial)} + \frac{1}{2}\nabla\widehat{\epsilon}^{[\geq 2]}. \quad (\text{S2.5})$$

For each of these subsystems, we show that the corresponding parts of k, σ are solutions.

Analysis of S0. The two functions $\delta^{[0]}, \sigma_{NN}^{[0]}$ are radial. Integration of (S0.1) along r with the trivial boundary condition $\delta^{[0]}|_{r=1} = 0$ directly leads to (4.30). (S0.2) is satisfied by (4.35). Therefore, $\delta^{[0]}$ and $\sigma_{NN}^{[0]}$ solve S0.

Analysis of S1. The equations (S1.1), (S1.2) are automatically satisfied by the definition of $\epsilon^{[1]}$ in (4.40). The same holds for (S1.4) by the definition of $*\sigma_N^{[1]}$ in (4.42). It remains to verify that (S1.3) is satisfied.

Applying $\mathrm{d}\!\!\!\diagup$ and $\mathrm{cu}\!\!\!\diagup$ to (S1.3), plugging in the definition of $\epsilon^{[1]}$ in (4.40) and using Lemma 2.35, we get that $\delta^{[1]}$ and $\sigma_{NN}^{[1]}$ must satisfy the compatibility conditions

$$\frac{1}{2}\not\Delta\delta^{[1]} + \partial_r^2\delta^{[1]} + \frac{7}{r}\partial_r\delta^{[1]} + \frac{9}{r^2}\delta^{[1]} = \frac{1}{r^4}\partial_r (r^4(\rho_N)^{[1]}) - \mathrm{d}\!\!\!\diagup\not\partial^{[1]}, \quad (4.47)$$

$$\frac{1}{r^4}\partial_r (r^4\sigma_{NN}^{[1]}) = \mathrm{cu}\!\!\!\diagup\not\partial^{[1]}. \quad (4.48)$$

With regard to the Hodge-Fourier decomposition, it holds that $\not\Delta\delta^{[1]} = -\frac{2}{r^2}\delta^{[1]}$. So, (4.47) is satisfied by the definition of $\delta^{[1]}$ in (4.31). Moreover, $\sigma_{NN}^{[1]}$ satisfies (4.48) by (4.36). Note that at the level of $l \geq 1$, \mathcal{P}_1 is a bijection, see Lemma 2.36, so the above shows that

(S1.3) is satisfied. To summarize, we showed that $\epsilon^{[1]}, \delta^{[1]}, \sigma_{NN}^{[1]}$ and $^*\sigma_N^{[1]}$ solve S1.

Analysis of S2. In the following, we use that $\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \Delta$. The equations (S2.1), (S2.2) and (S2.4) are satisfied in view of (4.40) and (4.46). It remains to prove (S2.3) and (S2.5). We start with (S2.3).

Applying div and curl to (S2.3) and using (4.40) leads to the compatibility conditions

$$\Delta \delta^{[\geq 2]} + \frac{4}{r} \partial_r \delta^{[\geq 2]} + \frac{6}{r^2} \delta^{[\geq 2]} = \frac{1}{r^3} \partial_r \left(r^3 (\rho_N)^{[\geq 2]} \right) - \text{div} \left(\frac{1}{2} \rho^{[\geq 2]} + ^*\sigma_N^{[\geq 2]} \right), \quad (4.49)$$

$$\frac{1}{r^3} \partial_r \left(r^3 \sigma_{NN}^{[\geq 2]} \right) = \text{curl} \left(\frac{1}{2} \rho^{[\geq 2]} + ^*\sigma_N^{[\geq 2]} \right). \quad (4.50)$$

The function $\delta^{[\geq 2]}$ defined in (4.32) satisfies (4.49) by the definition of $^*\sigma_{N_E}^{[\geq 2]}$ in (4.43) and the fact that

$$\text{div} \left(\frac{1}{2} \rho_H^{[\geq 2]} + ^*\sigma_{N_H}^{[\geq 2]} \right) = 0,$$

see the construction of $H^{(lm)}$ in (2.10). Furthermore, the $^*\sigma_{N_H}^{[\geq 2]}$ defined in (4.44) satisfies (4.50) because

$$\text{curl} \left(\frac{1}{2} \rho_E^{[\geq 2]} + ^*\sigma_{N_E}^{[\geq 2]} \right) = 0$$

by the construction of $E^{(lm)}$ in (2.10). This shows that (S2.3) is satisfied.

We turn now to (S2.5). Applying the divergence operator \mathcal{D}_2 to (S2.5) and using (4.46) leads to

$$\mathcal{D}_2 \left(^*(\widehat{\phi}) + \frac{1}{2} \nabla \otimes \epsilon^{[\geq 2]} \right) = \frac{1}{r^2} \nabla_N \left(r^2 \left(\frac{1}{2} \rho^{[\geq 2]} - ^*\sigma_N^{[\geq 2]} - \frac{1}{2} \nabla \delta^{[\geq 2]} - \frac{1}{r} \epsilon^{[\geq 2]} \right) \right). \quad (4.51)$$

This coincides with the definition of $^*(\widehat{\phi})$ in (4.45) and thus shows that (S2.5) is satisfied.

To summarise, we showed that $\delta^{[\geq 2]}, \epsilon^{[\geq 2]}, \sigma_{NN}^{[\geq 2]}, ^*\sigma_N^{[\geq 2]}, ^*(\widehat{\phi})^{[\geq 2]}, \hat{\eta}$ solve S2. This finishes the proof of Lemma 4.10. \square

We continue by controlling the regularity and boundary behaviour at S_1 of ζ_E, ζ_H and $\delta^{[\geq 2]}, \sigma_{NN}^{[\geq 2]}$.

Proposition 4.11. *Let $w \geq 2$ be an integer. Let $\rho = (\rho_N, \rho) \in \overline{\mathcal{H}}_{-5/2}^{w-2}$ be given. Let ζ_E, ζ_H be the vectorfields defined in (4.33)-(4.34), (4.38)-(4.39), and $\delta^{[\geq 2]}, \sigma_{NN}^{[\geq 2]}$ be the solutions to the elliptic PDEs (4.32), (4.37), respectively. Then, the following holds.*

- (1) **Regularity and boundary behaviour of ζ_E, ζ_H .** *The vectorfields ζ_E and ζ_H satisfy*

$$\|\zeta_E\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}} \lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}}, \quad \|\zeta_H\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}} \lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}}. \quad (4.52)$$

Moreover, for each $l \geq 2$, $m \in \{-l, \dots, l\}$, $\zeta_E^{(lm)}$ and $\zeta_H^{(lm)}$ satisfy

$$\begin{aligned} \int_1^\infty r^{1-l} \left(\zeta_E^{(lm)} - \left(\frac{l}{\sqrt{l(l+1)}} (\rho_N)^{(lm)} - \rho_E^{(lm)} \right) \right) dr &= 0, \\ \int_1^\infty r^{-1-\sqrt{l(l+1)+4}} \left(\zeta_H^{(lm)} + \rho_H^{(lm)} \right) dr &= 0. \end{aligned} \quad (4.53)$$

- (2) **Precise estimate for ζ_E and ζ_H .** *It holds that*

$$\begin{aligned} \|\mathcal{P}_2^{-1}(\nabla_N \zeta_E)\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}, \\ \|\mathcal{P}_2^{-1}(\nabla_N \zeta_H)\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}, \end{aligned} \quad (4.54)$$

with $\mathcal{P}_2^{-1}(\nabla_N \zeta_E), \mathcal{P}_2^{-1}(\nabla_N \zeta_H) \in \overline{\mathcal{H}}_{-5/2}^{w-2}$.

- (3) **Elliptic regularity and boundary behaviour of $\delta^{[\geq 2]}, \sigma_{NN}^{[\geq 2]}$.** *It holds that*

$$\begin{aligned} \|\delta^{[\geq 2]}\|_{\mathcal{H}_{-3/2}^{w-1}(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}, \\ \|\sigma_{NN}^{[\geq 2]}\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}. \end{aligned} \quad (4.55)$$

Furthermore,

$$\delta^{[\geq 2]} \in \overline{\mathcal{H}}_{-3/2}^{w-1}, \sigma_{NN}^{[\geq 2]} \in \overline{\mathcal{H}}_{-5/2}^{w-2}.$$

In particular, for $w > 2$,

$$\partial_r \delta^{[\geq 2]} \Big|_{r=1} = 0, \quad (4.56)$$

and for $w > 3$,

$$\partial_r \sigma_{NN}^{[\geq 2]} \Big|_{r=1} = 0. \quad (4.57)$$

- (4) **Precise estimate for $\partial_r \sigma_{NN}^{[\geq 2]}$.** *It holds that*

$$\left\| \mathcal{P}_1^{-1}(0, \partial_r \sigma_{NN}^{[\geq 2]}) \right\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}. \quad (4.58)$$

and moreover, $\mathcal{P}_1^{-1}(0, \partial_r \sigma_{NN}^{[\geq 2]}) \in \overline{\mathcal{H}}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})$.

Here $\mathcal{P}_1^{-1}, \mathcal{P}_2^{-1}$ denote the inverse operators to the elliptic $\mathcal{P}_1, \mathcal{P}_2$ on $(S_r, \overset{\circ}{\gamma})$, respectively, see Lemma 2.36.

Remark 4.12. *The quantities $\delta^{[\geq 2]}, \sigma_{NN}^{[\geq 2]}$ are solutions to the elliptic equations (4.32), (4.37) on $\mathbb{R}^3 \setminus \overline{B_1}$, respectively. Therefore their boundary regularity at S_1 is harder to estimate than for the other components of k and σ which all satisfy first order transport equations in r or Hodge systems on S_r .*

Proof. We first analyse ζ_E and ζ_H .

(1) Regularity and boundary behaviour of ζ_E, ζ_H . We begin by showing that the constants $c_E^{(lm)}, c_H^{(lm)}$ in (4.34), (4.39) are well-defined. By Cauchy-Schwarz, for all $l \geq 2$, $m \in \{-l, \dots, l\}$,

$$\begin{aligned} |c_E^{(lm)}| &= \left| \int_1^\infty r^{-l+1} \left(\frac{l}{\sqrt{l(l+1)}} (\rho_N)^{(lm)} - \rho_E^{(lm)} \right) dr \right| \\ &\leq \left(\int_1^\infty r^{-2l} dr \right)^{1/2} \left(\int_1^\infty r^2 \left(\frac{l}{\sqrt{l(l+1)}} (\rho_N)^{(lm)} - \rho_E^{(lm)} \right)^2 dr \right)^{1/2} \\ &\lesssim \frac{1}{\sqrt{2l-1}} \left(\int_1^\infty \left(r(\rho_N)^{(lm)} \right)^2 dr + \int_1^\infty \left(r\rho_E^{(lm)} \right)^2 dr \right)^{1/2}, \end{aligned} \quad (4.59)$$

and also for all $l \geq 2$,

$$\begin{aligned} |c_H^{(lm)}| &= \left| \int_1^\infty r^{-1-\sqrt{l(l+1)+4}} \rho_H^{(lm)} dr \right| \\ &\lesssim \left(\frac{1}{2\sqrt{l(l+1)+4}+3} \right)^{1/2} \left(\int_1^\infty \left(r\rho_H^{(lm)} \right)^2 dr \right)^{1/2}. \end{aligned} \quad (4.60)$$

We show next that for all integers $w \geq 2$,

$$\begin{aligned} \|\zeta_E\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}, \\ \|\zeta_H\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}. \end{aligned} \quad (4.61)$$

Consider first the case $w = 2$ of (4.61) for ζ_E , that is,

$$\|\zeta_E\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}. \quad (4.62)$$

By (4.33) and (4.38), for all $l \geq 2$, $\zeta_E^{(lm)}$ and $\zeta_H^{(lm)}$ vanish outside the interval $(1, 1 + 1/l)$. Therefore, for $l \geq 2$,

$$\begin{aligned}
\int_1^\infty \left(r \zeta_E^{(lm)} \right)^2 dr &= \left(c_E^{(lm)} \right)^2 l^2 \int_1^{1+1/l} r^{2l} \left((\partial_r \chi)(l(r-1)) \right)^2 dr \\
&\lesssim \left(c_E^{(lm)} \right)^2 \frac{l^2}{2l+1} \left[\left(1 + \frac{1}{l} \right)^{2l+1} - 1 \right] \\
&\lesssim \left(\int_1^\infty \left(r (\rho_N)^{(lm)} \right)^2 dr + \int_1^\infty \left(r \rho_E^{(lm)} \right)^2 dr \right),
\end{aligned} \tag{4.63}$$

where we uniformly estimated $\partial_r \chi$ and used (4.59) in the last step. Similarly, for $l \geq 2$, using (4.60),

$$\begin{aligned}
\int_1^\infty (r \zeta_H^{(lm)})^2 dr &= \left(c_H^{(lm)} \right)^2 \int_1^\infty r^{2+2\sqrt{l(l+1)+4}} (\partial_r (\chi(l(r-1))))^2 dr \\
&= \left(c_H^{(lm)} \right)^2 l^2 \int_1^{1+1/l} r^{2+2\sqrt{l(l+1)+4}} (\partial_r \chi)^2 (l(r-1)) dr \\
&\lesssim \left(c_H^{(lm)} \right)^2 l^2 \left(\frac{1}{3+2\sqrt{l(l+1)+4}} \right) \left(\left(1 + \frac{1}{l} \right)^{3+2\sqrt{l(l+1)+4}} - 1 \right) \\
&\lesssim \int_1^\infty \left(r \rho_H^{(lm)} \right)^2 dr.
\end{aligned}$$

Summing over l and m proves (4.62).

The case $w > 2$ of (4.61) for ζ_E is derived as follows. By Proposition 2.34, we improve (4.59) as follows,

$$\begin{aligned}
|c_E^{(lm)}| &\lesssim \frac{1}{\sqrt{2l-1}} \left(\int_1^\infty \left(r(\rho_N)^{(lm)} \right)^2 dr + \int_1^\infty \left(r\phi_E^{(lm)} \right)^2 dr \right)^{1/2} \\
&\lesssim \frac{1}{\sqrt{2l-1}\sqrt{l(l+1)}^{w-2}} \left((l(l+1))^{w-2} \int_1^\infty \left(r(\rho_N)^{(lm)} \right)^2 dr \right)^{1/2} \\
&\quad + \frac{1}{\sqrt{2l-1}\sqrt{l(l+1)}^{w-2}} \left((l(l+1))^{w-2} \int_1^\infty \left(r\phi_E^{(lm)} \right)^2 dr \right)^{1/2} \\
&\lesssim \frac{1}{\sqrt{2l-1}\sqrt{l(l+1)}^{w-2}} \left(\int_1^\infty r^{2(w-2)} \left(\frac{l(l+1)}{r^2} \right)^{w-2} \left(r(\rho_N)^{(lm)} \right)^2 dr \right)^{1/2} \\
&\quad + \frac{1}{\sqrt{2l-1}\sqrt{l(l+1)}^{w-2}} \left(\int_1^\infty r^{2(w-2)} \left(\frac{l(l+1)}{r^2} \right)^{w-2} \left(r\phi_E^{(lm)} \right)^2 dr \right)^{1/2}.
\end{aligned} \tag{4.64}$$

The terms in brackets on the right-hand side, in view of Proposition 2.34, correspond after summing over l, m to the $\overline{\mathcal{H}}_{-5/2}^{w-2}$ -norm of ρ which is bounded. These terms are therefore in particular summable.

On the other hand, we can explicitly calculate

$$\begin{aligned}
\partial_r \left(\zeta_E^{(lm)} \right) &= c_E^{(lm)} \frac{l-1}{r} r^{l-1} \partial_r \left(\chi(l(r-1)) \right) + c_E^{(lm)} r^{l-1} l \partial_r \left((\partial_r \chi)(l(r-1)) \right) \\
&\approx \frac{l}{r} \zeta_E^{(lm)}, \\
(\operatorname{div} \zeta_E)^{(lm)} &= -\frac{\sqrt{l(l+1)}}{r} \zeta_E^{(lm)}, \quad (\operatorname{curl} \zeta_E)^{(lm)} = 0.
\end{aligned} \tag{4.65}$$

Combining (4.64), (4.65) and using Propositions 2.30 and 2.34 and Lemma 2.35, we can estimate the derivatives of ζ_E similarly as in (4.62). This proves (4.61) for ζ_E for all $w \geq 2$. The estimates (4.61) for ζ_H are derived analogously and left to the reader. This proves (4.61) for all $w \geq 2$.

We next show that $\zeta_E, \zeta_H \in \overline{\mathcal{H}}_{-5/2}^{w-2}$ by proving that there exist sequences $(\zeta_E)_n, (\zeta_H)_n$ of smooth vectorfields with

$$\operatorname{supp}(\zeta_E)_n, \operatorname{supp}(\zeta_H)_n \subset \subset \mathbb{R}^3 \setminus \overline{B_1} \tag{4.66}$$

that converge as $n \rightarrow \infty$ in $\mathcal{H}_{-5/2}^{w-2}$ to

$$(\zeta_E)_n \rightarrow \zeta_E, (\zeta_H)_n \rightarrow \zeta_H. \quad (4.67)$$

Indeed, let

$$\begin{aligned} (\zeta_E)_n &:= \sum_{l=2}^n \sum_{m=-l}^l \zeta_E^{(lm)} E^{(lm)}, \\ (\zeta_H)_n &:= \sum_{l=2}^n \sum_{m=-l}^l \zeta_H^{(lm)} E^{(lm)}. \end{aligned}$$

By (2.1), (4.33) and (4.38), it follows that for each n , these are smooth vectorfields satisfying (4.66). By (4.61), the convergence (4.67) follows.

Next, we prove the integral identities (4.53). The first one follows by

$$\begin{aligned} \int_1^\infty r^{1-l} \zeta_E^{(lm)} dr &= \int_1^\infty c_E^{(lm)} \partial_r (\chi((r-1)l)) dr \\ &= c_E^{(lm)} [\chi((r-1)l)]_1^\infty \\ &= c_E^{(lm)} (1-0) \\ &= \int_1^\infty r^{1-l} \left(\frac{l}{\sqrt{l(l+1)}} (\rho_N)^{(lm)} - \rho_E^{(lm)} \right) dr. \end{aligned} \quad (4.68)$$

The second identity is proven similarly and left to the reader. This proves part (1) of Proposition 4.11.

(2) Precise estimate for ζ_E and ζ_H . We first prove (4.54). Consider the case $w = 2$ of the estimate for ζ_E in (4.54). Using the Hodge-Fourier formalism, see Proposition 2.34 and Lemmas 2.35 and 2.36, it suffices to prove

$$\int_1^\infty r^2 \left(\frac{r}{\sqrt{\frac{1}{2}l(l+1)-1}} \partial_r \zeta_E^{(lm)} \right)^2 dr \lesssim \int_1^\infty \left(r \rho_E^{(lm)} \right)^2 + \left(r \rho_H^{(lm)} \right)^2 + \left(r (\rho_N)^{(lm)} \right)^2 dr.$$

By (4.65), we can estimate

$$\begin{aligned}
& \int_1^\infty r^2 \left(\frac{r}{\sqrt{\frac{1}{2}l(l+1)-1}} \partial_r \zeta_E^{(lm)} \right)^2 dr \\
& \lesssim \int_1^\infty r^2 \left(\zeta_E^{(lm)} \right)^2 dr + \left(c_E^{(lm)} \right)^2 \int_1^\infty r^{2l+2} l^2 \left((\partial^2 \chi)(l(r-1)) \right)^2 dr \\
& \lesssim \int_1^\infty r^2 \left(\zeta_E^{(lm)} \right)^2 dr + l^2 \left(c_E^{(lm)} \right)^2 \int_1^{1+\frac{1}{l}} r^{2l+2} dr \\
& \lesssim \int_1^\infty r^2 \left(\zeta_E^{(lm)} \right)^2 dr + l^2 \left(c_E^{(lm)} \right)^2 \frac{1}{2l+3} \left(\left(1 + \frac{1}{l} \right)^{3+2l} - 1 \right) \\
& \lesssim \int_1^\infty r^2 \left(\zeta_E^{(lm)} \right)^2 dr + \frac{l^2}{2l+3} \left(c_E^{(lm)} \right)^2 \\
& \lesssim \int_1^\infty r^2 \left(\zeta_E^{(lm)} \right)^2 dr + \int_1^\infty \left(r \phi_E^{(lm)} \right)^2 + \left(r (\rho_N)^{(lm)} \right)^2 dr,
\end{aligned} \tag{4.69}$$

where we uniformly estimated $\partial_r^2 \chi$, used the fact that $\text{supp } \partial_r^2 \chi \subset [1, 1 + 1/l]$ and (4.59). Together with Proposition 4.11, this proves (4.54) for $w = 2$.

Consider now the case $w > 2$ of (4.54). On the one hand, by the higher regularity of $\phi_E^{(lm)}, (\rho_N)^{(lm)}$, the estimate of $c_E^{(lm)}$ improves, see (4.64). On the other hand, we can differentiate the explicit formula (4.33) by ∂_r , while taking angular derivatives correspond to multiplications by $\frac{\sqrt{l(l+1)}}{r}$. All terms appearing can be bounded analogously as in (4.69) by using the improved bounds for $c^{(lm)}$ and the fact that for all $n \geq 1$,

$$\text{supp } \partial_r^n \chi \subset [1, 1 + 1/l].$$

This proves (4.54) for ζ_E for $w \geq 2$. The proof for ζ_H is analogous and left to the reader.

It remains to show that $\mathcal{P}_2^{-1}(\nabla_N \zeta_E), \mathcal{P}_2^{-1}(\nabla_N \zeta_E) \in \overline{\mathcal{H}}_{-5/2}^{w-2}$. Consider the statement for $\mathcal{P}_2^{-1}(\nabla_N \zeta_E)$. For each $n \geq 2$, the smooth S_r -tangent tracefree symmetric 2-tensor

$$V_n := \sum_{l=2}^n \sum_{m=-l}^l \left(\mathcal{P}_2^{-1}(\nabla_N \zeta_E) \right)_\psi^{(lm)} \psi^{(lm)}$$

has compact support in $\mathbb{R}^3 \setminus \overline{B_1}$ by the definition of ζ_E in (4.33), see Lemma 2.33. Further, by (4.54), $V_n \rightarrow \mathcal{P}_2^{-1}(\nabla_N \zeta_E)$ as $n \rightarrow \infty$ in $\mathcal{H}_{-5/2}^{w-2}$. By definition of $\overline{\mathcal{H}}_{-5/2}^{w-2}$, see Definition 2.12, the statement for $\mathcal{P}_2^{-1}(\nabla_N \zeta_E)$ follows. The statement for $\mathcal{P}_2^{-1}(\nabla_N \zeta_H)$ follows analogously. This finishes the proof of part (2) of Proposition 4.11.

(3) Elliptic regularity and boundary behaviour of $\delta^{[\geq 2]}, \sigma_{NN}^{[\geq 2]}$. First, by the elliptic theory of Appendix C, it follows that for integers $w \geq 2$,

$$\delta^{[\geq 2]} \in \overline{H}^1 \cap H_{-3/2}^{w-1}(\mathbb{R}^3 \setminus \overline{B_1}), \quad \sigma_{NN}^{[\geq 2]} \in H_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})$$

with estimates

$$\begin{aligned} \|\delta^{[\geq 2]}\|_{H_{-3/2}^{w-1}(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}, \\ \|\sigma_{NN}^{[\geq 2]}\|_{H_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}. \end{aligned} \tag{4.70}$$

Indeed, $\delta^{[\geq 2]}$ is estimated in $\overline{H}_{-3/2}^1$ by Proposition C.4 and $\sigma_{NN}^{[\geq 2]}$ in $H_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})$ by Lemma C.11. Higher order regularity follows from Proposition C.6. The corresponding estimates obtained for $\delta^{[\geq 2]}$ and $\sigma_{NN}^{[\geq 2]}$ are in terms of norms of the right-hand sides of (4.32) and (4.37). In turn, these right-hand sides are estimated thanks to Corollary C.13 and the estimates of the part (1) of the proof for ζ_E and ζ_H .

We demonstrate now the improved boundary behaviour

$$\delta^{[\geq 2]} \in \overline{H}_{-3/2}^{w-1}, \quad \sigma_{NN}^{[\geq 2]} \in \overline{H}_{-5/2}^{w-2}.$$

We only need to consider the cases $w > 2$ for $\delta^{[\geq 2]}$ and $w > 3$ for $\sigma_{NN}^{[\geq 2]}$. Indeed, else the trivial extension to B_1 is regular and in view of the boundary conditions

$$\delta^{[\geq 2]}|_{r=1} = \sigma_{NN}^{[\geq 2]} = 0,$$

the statement follows by Proposition 2.13.

By Proposition C.9, it suffices to prove the following claim.

Claim 4.13. *If $w > 2$, then it holds that*

$$\partial_r \delta^{[\geq 2]}|_{r=1} = 0, \tag{4.71}$$

and if $w > 3$, then

$$\partial_r \sigma_{NN}^{[\geq 2]}|_{r=1} = 0. \tag{4.72}$$

First, by (4.70) it holds that for all $l \geq 2$, $m \in \{-l, \dots, l\}$, if $w > 2$, $w > 3$, respectively,

$$\begin{aligned} \int_1^\infty (\delta^{(lm)})^2 dr, \int_1^\infty (1+r)^2 (\partial_r \delta^{(lm)})^2 dr, \int_1^\infty (1+r)^4 (\partial_r^2 \delta^{(lm)})^2 dr < \infty, \\ \int_1^\infty (1+r)^2 (\sigma_{NN}^{(lm)})^2 dr, \int_1^\infty (1+r)^4 (\partial_r \sigma_{NN}^{(lm)})^2 dr, \int_1^\infty (1+r)^6 (\partial_r^2 \sigma_{NN}^{(lm)})^2 dr < \infty. \end{aligned}$$

By Lemma 2.14, it follows that

$$\begin{aligned} \sup_{r \in (1, \infty)} (1+r)^{1/2} \delta^{(lm)}, \sup_{r \in (1, \infty)} (1+r)^{3/2} \partial_r \delta^{(lm)} < \infty, \\ \sup_{r \in (1, \infty)} (1+r)^{3/2} \sigma_{NN}^{(lm)}, \sup_{r \in (1, \infty)} (1+r)^{5/2} \partial_r \sigma_{NN}^{(lm)} < \infty. \end{aligned} \quad (4.73)$$

We show now that if $w > 2$, then for all $l \geq 2$, $m \in \{-l, \dots, l\}$,

$$\partial_r \delta^{(lm)}|_{r=1} = 0.$$

Definition (4.32) is in the Hodge-Fourier formalism equivalent to the following ODEs on $r \in (1, \infty)$ for $\delta^{(lm)}$ with $l \geq 2$, $m \in \{-l, \dots, l\}$, see Lemma 2.33,

$$r^{l-2} \partial_r (r^{-2l} \partial_r (r^{l+2} \delta^{(lm)})) = \frac{1}{r^2} \partial_r (r^2 (\rho_N)^{(lm)}) - \frac{\sqrt{l(l+1)}}{r} (\not{\rho}_E^{(lm)} + \zeta_E^{(lm)}). \quad (4.74)$$

On the one hand, using that $\delta^{(lm)}|_{r=1} = 0$, $l \geq 2$, and (4.73), we get

$$\begin{aligned} \int_1^\infty \partial_r (r^{-2l} \partial_r (r^{l+2} \delta^{(lm)})) dr &= [r^{2-l} \partial_r \delta^{(lm)} + (l+2) r^{1-l} \delta^{(lm)}]_1^\infty \\ &= -\partial_r \delta^{(lm)}|_{r=1}. \end{aligned}$$

On the other hand, by (4.74),

$$\begin{aligned} \int_1^\infty \partial_r (r^{-2l} \partial_r (r^{l+2} \delta^{(lm)})) &= \int_1^\infty r^{-l} \partial_r \left(r^2 (\rho_N)^{(lm)} - \sqrt{l(l+1)} r^{1-l} (\not{\rho}_E^{(lm)} + \zeta_E^{(lm)}) \right) \\ &= \underbrace{\left[r^{2-l} (\rho_N)^{(lm)} \right]_1^\infty}_{=0} \\ &\quad + l \int_1^\infty r^{1-l} \left((\rho_N)^{(lm)} - \frac{\sqrt{l(l+1)}}{l} (\not{\rho}_E^{(lm)} + \zeta_E^{(lm)}) \right) \\ &= 0, \end{aligned} \quad (4.75)$$

where the boundary term vanished because $\rho_N \in \overline{H}_{-5/2}^{w-2}$, and where we also used the integral identity (4.53). This shows that

$$\partial_r \delta^{(lm)}|_{r=1} = 0$$

for all $l \geq 2$, $m \in \{-l, \dots, l\}$ and proves (4.71).

We show now that if $w > 3$, then for all $l \geq 2$, $m \in \{-l, \dots, l\}$,

$$\partial_r \sigma_{NN}^{(lm)}|_{r=1} = 0.$$

Definition (4.37) is in the Hodge-Fourier formalism equivalent to the following ODEs on $r \in (1, \infty)$ for $\sigma_{NN}^{(lm)}$ with $l \geq 2$, $m \in \{-l, \dots, l\}$, see Lemma 2.33,

$$r^{\sqrt{l(l+1)+4}-1} \partial_r \left(r^{1-2\sqrt{l(l+1)+4}} \partial_r \left(r^{\sqrt{l(l+1)+4}} \sigma_{NN}^{(lm)} \right) \right) = r \partial_r \left(\frac{\sqrt{l(l+1)}}{r^2} \left(\rho_H^{(lm)} + \zeta_H^{(lm)} \right) \right). \quad (4.76)$$

On the one hand, using that $\sigma_{NN}^{(lm)}|_{r=1} = 0$, $l \geq 2$ and (4.73),

$$\begin{aligned} \int_1^\infty \partial_r \left(r^{1-2\sqrt{l(l+1)+4}} \partial_r \left(r^{\sqrt{l(l+1)+4}} \sigma_{NN}^{(lm)} \right) \right) &= \left[\sqrt{l(l+1)+4} r^{-\sqrt{l(l+1)+4}} \sigma_{NN}^{(lm)} \right]_1^\infty \\ &\quad + \left[r^{1-\sqrt{l(l+1)+4}} \partial_r \sigma_{NN}^{(lm)} \right]_1^\infty \\ &= -\partial_r \sigma_{NN}^{(lm)}|_{r=1}. \end{aligned}$$

On the other hand, using (4.76),

$$\begin{aligned} &\int_1^\infty \partial_r \left(r^{1-2\sqrt{l(l+1)+4}} \partial_r \left(r^{\sqrt{l(l+1)+4}} \sigma_{NN}^{(lm)} \right) \right) \\ &= \int_1^\infty r^{2-\sqrt{l(l+1)+4}} \partial_r \left(\frac{\sqrt{l(l+1)}}{r^2} \left(\rho_H^{(lm)} + \zeta_H^{(lm)} \right) \right) \\ &= \left[r^{-\sqrt{l(l+1)+4}} \sqrt{l(l+1)} \left(\rho_H^{(lm)} + \zeta_H^{(lm)} \right) \right]_1^\infty \\ &\quad + \sqrt{l(l+1)} \left(\sqrt{l(l+1)+4} - 2 \right) \int_1^\infty r^{-\sqrt{l(l+1)+4}-1} \left(\rho_H^{(lm)} + \zeta_H^{(lm)} \right) \\ &= 0, \end{aligned}$$

where we used the integral identity (4.53) and the boundary term vanished because $\rho, \zeta_H \in \overline{\mathcal{H}}_{-5/2}^{w-2}$. This shows that for all $l \geq 2$, $m \in \{-l, \dots, l\}$,

$$\partial_r \sigma_{NN}^{(lm)}|_{r=1} = 0$$

and proves (4.72). This finishes the proof of Claim 4.13. Hence, we have obtained the control of $\delta^{[\geq 2]} \in \overline{\mathcal{H}}_{-3/2}^{w-1}$, $\sigma_{NN}^{[\geq 2]} \in \overline{\mathcal{H}}_{-5/2}^{w-2}$. This finishes the proof of part (3) of Proposition 4.11.

Remark 4.14. For $l \geq 2$, $m \in \{-l, \dots, l\}$, if $\rho_H^{(lm)}$ is compactly supported in $\mathbb{R}^3 \setminus \overline{B_1}$, then $\sigma_{NN}^{(lm)}$ is compactly supported in $\mathbb{R}^3 \setminus \overline{B_1}$. Indeed, this follows by integrating the radial ODE (4.76) and using that by the construction of $\zeta_H^{(lm)}$, $\text{supp} \zeta_H^{(lm)} \subset\subset \mathbb{R}^3 \setminus \overline{B_1}$ and $\partial_r \sigma_{NN}^{(lm)}|_{r=1} = 0$.

(4) Precise estimate for $\partial_r \sigma_{NN}^{[\geq 2]}$. First consider the case $w = 2$ of (4.58), that is,

$$\left\| \mathcal{D}_1^{-1}(0, \partial_r \sigma_{NN}^{[\geq 2]}) \right\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}. \quad (4.77)$$

Using the Fourier-Hodge formalism and the previous estimates for $\sigma_{NN}^{[\geq 2]}, \zeta_H$, it suffices to prove

$$\int_1^\infty r^2 \left(\frac{r}{\sqrt{l(l+1)}} \partial_r \sigma_{NN}^{(lm)} \right)^2 dr \lesssim \int_1^\infty \left(r \rho_H^{(lm)} \right)^2 + \left(r \sigma_{NN}^{(lm)} \right)^2 + \left(r \zeta_H^{(lm)} \right)^2 dr. \quad (4.78)$$

First, we rewrite the integrand by using (4.76). Indeed, multiplying (4.76) by $r^{-\sqrt{l(l+1)+4}+1}$ and integrating from 1 to $r \geq 1$ leads, after integration by parts, to the expression

$$\begin{aligned} & \frac{r}{\sqrt{l(l+1)}} \partial_r \sigma_{NN}^{(lm)} \\ &= - \frac{\sqrt{l(l+1)+4}}{\sqrt{l(l+1)}} \sigma_{NN}^{(lm)} + \left(\rho_H^{(lm)} + \zeta_H^{(lm)} \right) \\ & \quad - \left(2 - \sqrt{l(l+1)+4} \right) r^{\sqrt{l(l+1)+4}} \left(\int_1^r (r')^{-\sqrt{l(l+1)+4}-1} \left(\rho_H^{(lm)} + \zeta_H^{(lm)} \right) dr' \right), \end{aligned} \quad (4.79)$$

where the boundary terms at $r = 1$ vanished because $\rho, \zeta_H \in \overline{\mathcal{H}}_{-5/2}^{w-2}$ and

$$\sigma_{NN}^{[\geq 2]}|_{r=1} = \partial_r \sigma_{NN}^{[\geq 2]}|_{r=1} = 0.$$

We are now in the position to prove (4.78). We have

$$\begin{aligned}
& \int_1^\infty r^2 \left(\frac{r}{\sqrt{l(l+1)}} \partial_r \sigma_{NN}^{(lm)} \right)^2 dr \\
& \lesssim \int_1^\infty r^2 \left[\left(\sigma_{NN}^{(lm)} \right)^2 + \left(\rho_H^{(lm)} \right)^2 + \left(\zeta_H^{(lm)} \right)^2 \right] dr \\
& \quad + \underbrace{(\sqrt{l(l+1)+4}-2)^2 \int_1^\infty r^{2\sqrt{l(l+1)+4}+2} \left(\int_1^r (r')^{-\sqrt{l(l+1)+4}-1} \left(\rho_H^{(lm)} + \zeta_H^{(lm)} \right) dr' \right)^2}_{:=I_2} dr.
\end{aligned} \tag{4.80}$$

By the integral identity (4.53) we can rewrite I_2 and use integration by parts to get

$$\begin{aligned}
I_2 &= \int_1^\infty r^{2\sqrt{l(l+1)+4}+2} \left(\int_r^\infty (r')^{-\sqrt{l(l+1)+4}-1} \left(\rho_H^{(lm)} + \zeta_H^{(lm)} \right) dr' \right)^2 dr \\
&= \frac{1}{2\sqrt{l(l+1)+4}+3} \left[r^{2\sqrt{l(l+1)+4}+3} \left(\int_r^\infty (r')^{-\sqrt{l(l+1)+4}-1} \left(\rho_H^{(lm)} + \zeta_H^{(lm)} \right) dr' \right)^2 \right]_1^\infty \\
&\quad + 2 \int_1^\infty \frac{r^{\sqrt{l(l+1)+4}+2} \left(\rho_H^{(lm)} + \zeta_H^{(lm)} \right)}{2\sqrt{l(l+1)+4}+3} \left(\int_r^\infty (r')^{-\sqrt{l(l+1)+4}-1} \left(\rho_H^{(lm)} + \zeta_H^{(lm)} \right) dr' \right) dr.
\end{aligned} \tag{4.81}$$

The boundary term on the right-hand side can be estimated as follows

$$\begin{aligned}
& \left| r^{2\sqrt{l(l+1)+4}+3} \left(\int_r^\infty (r')^{-\sqrt{l(l+1)+4}-1} \left(\rho_H^{(lm)} + \zeta_H^{(lm)} \right) dr' \right)^2 \right| \\
& \leq r^{2\sqrt{l(l+1)+4}+3} \left(\int_r^\infty (r')^{-2\sqrt{l(l+1)+4}-4} dr' \right) \left(\int_1^\infty \left(r \rho_H^{(lm)} \right)^2 + \left(r \zeta_H^{(lm)} \right)^2 dr \right) \\
& \leq \frac{1}{2\sqrt{l(l+1)+4}+3} \left(\int_1^\infty \left(r \rho_H^{(lm)} \right)^2 + \left(r \zeta_H^{(lm)} \right)^2 dr \right).
\end{aligned}$$

The integral term on the right-hand side of (4.81) is estimated by Cauchy-Schwarz as

$$\begin{aligned} & \int_1^\infty r^{2\sqrt{l(l+1)+4}+2} \left(\phi_H^{(lm)} + \zeta_H^{(lm)} \right) \left(\int_1^r (r')^{-\sqrt{l(l+1)+4}-1} \left(\phi_H^{(lm)} + \zeta_H^{(lm)} \right) dr' \right) dr \\ & \leq (I_2)^{1/2} \left(\int_1^\infty \left(r \phi_H^{(lm)} \right)^2 + \left(r \zeta_H^{(lm)} \right)^2 dr \right)^{1/2}. \end{aligned}$$

Putting everything together, we arrive at

$$I_2 \lesssim \frac{1}{(2\sqrt{l(l+1)+4}+3)^2} \left(\int_1^\infty \left(r \phi_H^{(lm)} \right)^2 + \left(r \zeta_H^{(lm)} \right)^2 dr \right).$$

Plugging this into (4.80) yields

$$\int_1^\infty r^2 \left(\frac{r}{\sqrt{l(l+1)}} \partial_r \sigma_{NN}^{(lm)} \right)^2 dr \lesssim \int_1^\infty \left(r \phi_H^{(lm)} \right)^2 + \left(r \sigma_{NN}^{(lm)} \right)^2 + \left(r \zeta_H^{(lm)} \right)^2 dr.$$

This proves (4.77), that is, the case $w = 2$ of (4.58).

We turn now to the case $w > 2$ of (4.58). By differentiating (4.79) in r or taking the tangential derivative ∇ which on the Fourier side amounts to multiplication by $\frac{\sqrt{l(l+1)}}{r}$, and using Proposition 2.34 and Lemma 2.35, we eventually get that for all $w \geq 2$,

$$\left\| \mathcal{P}_1^{-1} \left(0, \partial_r \sigma_{NN}^{[\geq 2]} \right) \right\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}, \quad (4.82)$$

that is, we proved (4.58).

It remains to show that $\mathcal{P}_1^{-1}(0, \partial_r \sigma_{NN}^{[\geq 2]}) \in \overline{\mathcal{H}}_{-5/2}^{w-2}$. By definition, it suffices to prove that there is a sequence X_n of smooth vectorfields on $\mathbb{R}^3 \setminus \overline{B_1}$ with

$$\text{supp } X_n \subset\subset \mathbb{R}^3 \setminus \overline{B_1}$$

that converges as $n \rightarrow \infty$ in $\mathcal{H}_{-5/2}^{w-2}$,

$$X_n \rightarrow \mathcal{P}_1^{-1} \left(0, \partial_r \sigma_{NN}^{[\geq 2]} \right).$$

Let ρ_n be a sequence of smooth vectorfields on $\mathbb{R}^3 \setminus \overline{B_1}$ such that for all n ,

$$\text{supp } \rho_n \subset\subset \mathbb{R}^3 \setminus \overline{B_1}$$

and as $n \rightarrow \infty$, in $\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})$,

$$\rho_n \rightarrow \rho.$$

Consider the sequence

$$\rho_n^{[\leq n]} := \sum_{l=1}^n \sum_{m=-l}^l \left((\rho_n)_E^{(lm)} E^{(lm)} + (\rho_n)_H^{(lm)} H^{(lm)} \right)$$

which satisfies for all n ,

$$\text{supp } \rho_n^{[\leq n]} \subset\subset \mathbb{R}^3 \setminus \overline{B_1}$$

and as $n \rightarrow \infty$, in $\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})$,

$$\rho_n^{[\leq n]} \rightarrow \rho.$$

By Remark 4.14 and the higher regularity estimates (4.52) and (4.70), it follows that solutions $(\sigma_{NN}^{[\geq 2]})_n$ to (4.37) with $\rho_n^{[\leq n]}$ and corresponding $(\zeta_H)_n$ defined in (4.38) on the right-hand side are smooth and satisfy

$$\text{supp } (\sigma_{NN}^{[\geq 2]})_n \subset\subset \mathbb{R}^3 \setminus \overline{B_1}.$$

This shows that

$$X_n := \mathcal{P}_1^{-1} \left(0, \partial_r (\sigma_{NN}^{[\geq 2]})_n \right),$$

is a sequence of smooth vectorfields with

$$\text{supp } X_n \subset\subset \mathbb{R}^3 \setminus \overline{B_1}.$$

Furthermore, by linearity and (4.58), as $n \rightarrow \infty$,

$$\left\| X_n - \mathcal{P}_1^{-1} \left(0, \partial_r \sigma_{NN}^{[\geq 2]} \right) \right\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|(\rho_n^{[\leq n]})_H^{[\geq 2]} - \rho_H^{[\geq 2]}\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}} \rightarrow 0.$$

The above implies that $\mathcal{P}_1^{-1}(0, \partial_r \sigma_{NN}^{[\geq 2]}) \in \overline{\mathcal{H}}_{-5/2}^{w-2}$. This finishes the control of $\mathcal{P}_1^{-1}(0, \partial_r \sigma_{NN}^{[\geq 2]})$ and hence concludes the proof of Proposition 4.11. \square

In the following lemma, we estimate all quantities that were not yet bounded in Proposition 4.11 and obtain the full regularity and boundary control of k and σ .

Lemma 4.15 (Full boundary control and regularity). *For $\rho = (\rho_N, \rho) \in \overline{\mathcal{H}}_{-5/2}^{w-2}$, the symmetric 2-tensors k and σ defined in (4.29)-(4.46) satisfy $k \in \overline{\mathcal{H}}_{-3/2}^{w-1}$, $\sigma \in \overline{\mathcal{H}}_{-5/2}^{w-2}$, with*

$$\begin{aligned} \|k\|_{\overline{\mathcal{H}}_{-3/2}^{w-1}} &\lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}}, \\ \|\sigma\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}} &\lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}}. \end{aligned} \tag{4.83}$$

Proof. In view of Lemma 2.19 and the decomposition of k and σ introduced at the beginning of Section 4.3.1, we prove that $\delta, \epsilon, \hat{\eta} \in \overline{\mathcal{H}}_{-3/2}^{w-1}$ and $\sigma_{NN}, {}^* \sigma_N, {}^* (\widehat{\phi}) \in \overline{\mathcal{H}}_{-5/2}^{w-2}$ together with quantitative estimates. We estimate the terms in the order they were introduced in (4.29)-(4.46).

Control of δ .

Control of $\delta^{[0]}$. First we show that for all $w \geq 2$,

$$\|\delta^{[0]}\|_{H_{-3/2}^{w-1}} \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}}. \quad (4.84)$$

First consider the case $w = 2$, that is,

$$\|\delta^{[0]}\|_{H_{-3/2}^1} \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^0}.$$

By (4.30) we can rewrite

$$\begin{aligned} \|\delta^{[0]}\|_{H_{-3/2}^0}^2 &= \int_1^\infty \int_{S_r} (\delta^{[0]})^2 dr = \int_1^\infty \frac{4\pi r^2}{r^6} \left(\int_1^r (r')^3 (\rho_N)^{[0]} dr' \right)^2 dr \\ &= \frac{1}{4\pi} \int_1^\infty \frac{1}{r^4} \left(\int_1^r r' \int_{S_{r'}} (\rho_N)^{[0]} dr' \right)^2 dr, \end{aligned}$$

where we used that $\delta^{[0]}$ and $(\rho_N)^{[0]}$ are radial. This expression allows us to estimate by partial integration

$$\begin{aligned} \|\delta^{[0]}\|_{H_{-3/2}^0}^2 &= \left[\left(-\frac{1}{3r^3} \right) \left(\int_1^r r' \int_{S_{r'}} (\rho_N)^{[0]} dr' \right)^2 \right]_1^\infty \\ &\quad + \frac{2}{3} \int_1^\infty \frac{1}{r^3} \left(r \int_{S_r} (\rho_N)^{[0]} \right) \left(\int_1^r r' \int_{S_{r'}} (\rho_N)^{[0]} dr' \right) dr \\ &\leq \frac{2}{3} \left(\int_1^\infty \left(\int_{S_r} (\rho_N)^{[0]} \right)^2 dr \right)^{1/2} \left(\int_1^\infty \frac{1}{r^4} \left(\int_1^r r' \int_{S_{r'}} (\rho_N)^{[0]} dr' \right)^2 dr \right)^{1/2} \quad (4.85) \\ &= \frac{2}{3} \left(\int_1^\infty \left(\int_{S_r} (\rho_N)^{[0]} \right)^2 dr \right)^{1/2} \|\delta^{[0]}\|_{H_{-3/2}^0} \end{aligned}$$

where the boundary term was discarded because it has non-positive sign. This implies that

$$\begin{aligned}
\|\delta^{[0]}\|_{H_{-3/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2 &\lesssim \int_1^\infty \left(\int_{S_r} (\rho_N)^{[0]} \right)^2 dr \\
&\lesssim \int_1^\infty \int_{S_r} \left(r (\rho_N)^{[0]} \right)^2 dr \\
&\lesssim \|(\rho_N)^{[0]}\|_{\overline{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2.
\end{aligned} \tag{4.86}$$

The radial derivative $\partial_r \delta^{[0]}$ equals by (4.30)

$$\partial_r \delta^{[0]} = -\frac{3}{r} \delta^{[0]} + (\rho_N)^{[0]}. \tag{4.87}$$

This yields with (4.86) the estimate

$$\|\partial_r \delta^{[0]}\|_{H_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|(\rho_N)^{[0]}\|_{H_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}. \tag{4.88}$$

The tangential derivative vanishes because $\delta^{[0]}$ is radial. This proves the case $w = 2$ of (4.84).

Consider now the case $w > 2$ of (4.84). Higher radial regularity follows by differentiating (4.87), and higher tangential regularity is trivial since $\delta^{[0]}$ is radial. This proves (4.84) for $w \geq 2$.

For the control of $\delta^{[0]}$ it remains to show that $\delta^{[0]} \in \overline{H}_{-3/2}^{w-1}$. Indeed, this follows by (4.87), the fact that $\rho \in \overline{\mathcal{H}}_{-5/2}^{w-2}$ and Proposition 2.13. This finishes the control of $\delta^{[0]}$.

Control of $\delta^{[1]}$. First we show that for all $w \geq 2$,

$$\|\delta^{[1]}\|_{H_{-3/2}^{w-1}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}. \tag{4.89}$$

Consider first the case $w = 2$, that is,

$$\|\delta^{[1]}\|_{H_{-3/2}^1(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}.$$

Integrating (4.31) yields the explicit form

$$\delta^{[1]} = \frac{1}{r^4} \int_1^r r' \underbrace{\left(\int_1^{r'} \left(\frac{1}{r''} \partial_r ((r'')^4 (\rho_N)^{[1]}) - (r'')^3 \operatorname{div} \rho^{[1]} \right) dr'' \right)}_{:= I_1(r')} dr'. \tag{4.90}$$

Using (4.90) and integration by parts in r , we estimate

$$\begin{aligned}
\|\delta^{[1]}\|_{H_{-3/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2 &= \int_1^\infty \int_{S_r} (\delta^{[1]})^2 dr \\
&= \int_1^\infty \int_{S_r} \frac{1}{r^8} \left(\int_1^r r' I_1(r') dr' \right)^2 dr \\
&= -\frac{1}{5} \left[\int_{S_r} \frac{1}{r^7} \left(\int_1^r r' I_1(r') dr' \right)^2 \right]_1^\infty \\
&\quad + \frac{2}{5} \int_1^\infty \int_{S_r} \frac{1}{r^7} (r I_1(r)) \left(\int_1^r r' I_1(r') dr' \right) dr \\
&\leq \frac{2}{5} \left(\int_1^\infty \int_{S_r} \frac{1}{r^8} \left(\int_1^r r' I_1(r') dr' \right)^2 dr \right)^{1/2} \left(\int_1^\infty \int_{S_r} \frac{1}{r^4} (I_1)^2(r) dr \right)^{1/2} \\
&= \frac{2}{5} \|\delta^{[1]}\|_{H_{-3/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \left(\int_1^\infty \int_{S_r} \frac{1}{r^4} (I_1)^2(r) dr \right)^{1/2},
\end{aligned}$$

where the boundary term was discarded because of its non-positive sign. This shows that

$$\|\delta^{[1]}\|_{H_{-3/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2 \lesssim \int_1^\infty \int_{S_r} \frac{1}{r^4} (I_1)^2(r) dr. \quad (4.91)$$

By a similar integration by parts, we further have

$$\int_1^\infty \int_{S_r} \frac{1}{r^4} (I_1)^2(r) dr \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2, \quad (4.92)$$

where we used that at $l = 1$,

$$\|\operatorname{div} \rho^{[1]}\|_{H_{-7/2}^0} \lesssim \|\rho^{[1]}\|_{H_{-5/2}^0}.$$

Together, (4.91) and (4.92) imply

$$\|\delta^{[1]}\|_{H_{-3/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}.$$

Moreover, by (4.90) the radial derivative $\partial_r \delta^{[1]}$ is

$$\partial_r \delta^{[1]} = -\frac{4}{r} \delta^{[1]} + \frac{1}{r^3} I_1(r).$$

Therefore (4.91) and (4.92) imply that

$$\|\partial_r \delta^{[1]}\|_{H_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2.$$

The tangential regularity of $\delta^{[1]}$ follows immediately from the fact that $l = 1$,

$$\|\nabla \delta^{[1]}\|_{H_{-5/2}^0} \lesssim \|\delta^{[1]}\|_{H_{-3/2}^0}.$$

This proves the case $w = 2$ of (4.89).

We turn now to the case $w > 2$ of (4.89). For higher radial regularity, differentiate the defining ODE (4.31),

$$\begin{cases} \partial_r^2 \delta^{[1]} + \frac{7}{r} \partial_r \delta^{[1]} + \frac{8}{r^2} \delta^{[1]} = \frac{1}{r^4} \partial_r \left(r^4 (\rho_N)^{[1]} \right) - \text{div} \rho^{[1]}, & \text{on } \mathbb{R}^3 \setminus \overline{B_1} \\ \delta^{[1]}|_{r=1} = \partial_r \delta^{[1]}|_{r=1} = 0. \end{cases} \quad (4.93)$$

Higher tangential regularity follows at the level of $l = 1$ in the Hodge-Fourier decomposition by the observation that for $n \geq 0$

$$\begin{aligned} \|\nabla^n \delta^{[1]}\|_{H_{-3/2-n}^0(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|\delta^{[1]}\|_{H_{-3/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \\ &\lesssim \|\rho\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}. \end{aligned} \quad (4.94)$$

This proves (4.89) for $w \geq 2$.

It remains to show that $\delta^{[1]} \in \overline{H}_{-3/2}^{w-1}$. Indeed, this follows by (4.93) and $\rho \in \overline{\mathcal{H}}_{-5/2}^{w-2}$ with Proposition 2.13. This finishes the control of $\delta^{[1]}$.

The full control of δ . Recall that

$$\delta = \delta^{[0]} + \delta^{[1]} + \delta^{[\geq 2]}.$$

Above we proved that for $w \geq 2$, $\delta^{[0]}, \delta^{[1]} \in \overline{H}_{-3/2}^{w-1}$ with the estimate

$$\|\delta^{[0]} + \delta^{[1]}\|_{H_{-3/2}^{w-1}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho\|_{\overline{\mathcal{H}}_{-3/2}^{w-2}}.$$

In Proposition 4.11, we proved that for $w \geq 2$, $\delta^{[\geq 2]} \in \overline{H}_{-3/2}^{w-1}$ with the estimate

$$\|\delta^{[\geq 2]}\|_{H_{-3/2}^{w-1}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho\|_{\overline{\mathcal{H}}_{-3/2}^{w-2}}.$$

Together this proves that for $w \geq 2$, $\delta \in \overline{H}_{-3/2}^{w-1}$ with the estimate

$$\|\delta\|_{H_{-3/2}^{w-1}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho\|_{\overline{\mathcal{H}}_{-3/2}^{w-2}}$$

and hence finishes the control of δ .

Control of σ_{NN} .

Control of $\sigma_{NN}^{[1]}$. First we show that for all $w \geq 2$,

$$\|\sigma_{NN}^{[1]}\|_{H_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho^{[1]}\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}. \quad (4.95)$$

Consider first the case $w = 2$, that is,

$$\|\sigma_{NN}^{[1]}\|_{H_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho^{[1]}\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}.$$

Recall (4.36),

$$\sigma_{NN}^{[1]} = \frac{1}{r^4} \int_1^r (r')^4 \operatorname{curl} \rho^{[1]} dr'.$$

Using this expression, the case $w = 2$ of (4.95) can be derived like for $\delta^{[0]}$ before, see (4.85) and (4.86). In particular, use that at $l = 1$,

$$\|\operatorname{curl} \rho^{[1]}\|_{H_{-7/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho^{[1]}\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}.$$

We turn now to the case $w > 2$ of (4.95). Higher radial regularity is proved by using and differentiating the defining ODE,

$$\begin{cases} \partial_r \sigma_{NN}^{[1]} + \frac{4}{r} \sigma_{NN}^{[1]} = \operatorname{curl} \rho^{[1]} \\ \sigma_{NN}^{[1]}|_{r=1} = 0. \end{cases} \quad (4.96)$$

Higher tangential regularity is automatic at $l = 1$, as in (4.94). This proves (4.95) for all $w \geq 2$.

It remains to show that $\sigma_{NN}^{[1]} \in \overline{H}_{-5/2}^{w-2}$. This follows by (4.96) and $\rho \in \overline{\mathcal{H}}_{-5/2}^{w-2}$ with Proposition 2.13. This finishes the control of $\sigma_{NN}^{[1]}$.

The full control of σ_{NN} . Recall that

$$\sigma_{NN} = \sigma_{NN}^{[1]} + \sigma_{NN}^{[\geq 2]}.$$

Above, we proved that for $w \geq 2$, $\sigma_{NN}^{[1]} \in \overline{H}_{-5/2}^{w-2}$ with the estimate

$$\|\sigma_{NN}^{[1]}\|_{\overline{H}_{-5/2}^{w-2}} \lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}}.$$

In Proposition 4.11, we proved that for $w \geq 2$, $\sigma_{NN}^{[\geq 2]} \in \overline{H}_{-5/2}^{w-2}$ with the estimate

$$\|\sigma_{NN}^{[\geq 2]}\|_{\overline{H}_{-5/2}^{w-2}} \lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}}.$$

Together this proves that for $w \geq 2$, $\sigma_{NN} \in \overline{H}_{-5/2}^{w-2}$ with the estimate

$$\|\sigma_{NN}\|_{\overline{H}_{-5/2}^{w-2}} \lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}}$$

and hence finishes the control of σ_{NN} .

Control of ϵ . First we show that for $w \geq 2$

$$\|\epsilon\|_{\mathcal{H}_{-3/2}^{w-1}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}. \quad (4.97)$$

Consider first the case $w = 2$ of (4.97),

$$\|\epsilon\|_{\mathcal{H}_{-3/2}^1(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}.$$

By (4.40) we have on S_r , for $r \geq 1$,

$$\mathcal{P}_1 \epsilon = \left(\rho_N - \frac{1}{r^3} \partial_r (r^3 \delta), \sigma_{NN} \right). \quad (4.98)$$

Proposition 2.23 and the estimates above for δ and σ_{NN} yield

$$\begin{aligned} & \|\epsilon\|_{\mathcal{H}_{-3/2}^2(\mathbb{R}^3 \setminus \overline{B_1})}^2 + \|\nabla \epsilon\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2 \\ & \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2 + \|\delta\|_{\mathcal{H}_{-3/2}^1(\mathbb{R}^3 \setminus \overline{B_1})}^2 + \|\sigma_{NN}\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2 \\ & \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2. \end{aligned}$$

For radial regularity, we use that by Lemma 4.10, ϵ also solves (S1.3) and (S2.3), together with (4.43) and (4.44) to obtain

$$\begin{aligned} \frac{1}{r^3} \nabla_N (r^3 \epsilon^{[1]}) &= \rho^{[1]} + \frac{1}{2} \nabla \delta^{[1]}, \\ \frac{1}{r^2} \nabla_N (r^2 \epsilon_E^{[\geq 2]}) &= \rho_E^{[\geq 2]} + \zeta_E + (\nabla \delta)_E^{[\geq 2]}, \\ \frac{1}{r^2} \nabla_N (r^2 \epsilon_H^{[\geq 2]}) &= \frac{1}{2} \rho_H^{[\geq 2]} + {}^* \sigma_{NH}^{[\geq 2]} \\ &= \mathcal{P}_1^{-1} \left(0, \frac{1}{r^3} \partial_r (r^3 \sigma_{NN}^{[\geq 2]}) \right). \end{aligned} \quad (4.99)$$

By Proposition 4.11 and the above estimates for δ , this yields the bounds

$$\|\nabla_N \epsilon\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}.$$

This proves the case $w = 2$ of (4.97).

Consider now the case $w > 2$ of (4.97). Higher tangential regularity is derived by tangentially differentiating (4.98) and using Propositions 2.24 and 4.11. Higher radial regularity follows by applying ∇_N to (4.99) and using Proposition 4.11. This proves (4.97) for all $w \geq 2$.

It remains to show that $\epsilon \in \overline{\mathcal{H}}_{-3/2}^{w-1}$. Indeed, this follows by (4.99) and the fact that $\rho, \mathcal{P}_1^{-1} \left(0, \frac{1}{r^3} \partial_r \left(r^3 \sigma_{NN}^{[\geq 2]} \right) \right), \zeta_E, \nabla \delta \in \overline{\mathcal{H}}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})$ together with Proposition 2.13. This finishes the control of ϵ .

Control of ${}^*\sigma_{\mathcal{N}}$. First we show that for all $w \geq 2$

$$\|{}^*\sigma_{\mathcal{N}}\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}. \quad (4.100)$$

This control of ${}^*\sigma_{\mathcal{N}}$ follows by the control of the previous quantities. Indeed, recall from (4.41)-(4.44),

$$\begin{aligned} {}^*\sigma_{\mathcal{N}}^{[1]} &= \frac{1}{2} \rho^{[1]} - \frac{1}{2} \nabla \delta^{[1]} - \frac{1}{r} \epsilon^{[1]}, \\ {}^*\sigma_{\mathcal{N}E}^{[\geq 2]} &= \frac{1}{2} \rho_E^{[\geq 2]} + \zeta_E, \\ {}^*\sigma_{\mathcal{N}H}^{[\geq 2]} &= -\frac{1}{2} \rho_H^{[\geq 2]} + \mathcal{P}_1^{-1} \left(0, \frac{1}{r^3} \partial_r \left(r^3 \sigma_{NN}^{[\geq 2]} \right) \right). \end{aligned} \quad (4.101)$$

This implies by Proposition 4.11 and the above control of δ and ϵ

$$\begin{aligned} \|{}^*\sigma_{\mathcal{N}}^{[1]}\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} + \|\delta\|_{H_{-3/2}^{w-1}} + \|\epsilon\|_{\mathcal{H}_{-3/2}^{w-1}(\mathbb{R}^3 \setminus \overline{B_1})} \\ &\lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}, \\ \|{}^*\sigma_{\mathcal{N}E}^{[\geq 2]}\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} + \|\zeta_E\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} \\ &\lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}, \\ \|{}^*\sigma_{\mathcal{N}H}^{[\geq 2]}\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} + \left\| \mathcal{P}_1^{-1} \left(0, \frac{1}{r^3} \partial_r \left(r^3 \sigma_{NN}^{[\geq 2]} \right) \right) \right\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} \\ &\lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}. \end{aligned}$$

This proves (4.100) for all $w \geq 2$.

It remains to show that ${}^*\sigma_{\mathcal{N}} \in \overline{\mathcal{H}}_{-5/2}^{w-2}$. This follows by (4.101) and $\delta, \epsilon \in \overline{\mathcal{H}}_{-3/2}^{w-1}$, $\rho, \zeta_E, \mathcal{P}_1^{-1} \left(0, \frac{1}{r^3} \partial_r \left(r^3 \sigma_{NN}^{[\geq 2]} \right) \right) \in \overline{\mathcal{H}}_{-5/2}^{w-2}$ together with Proposition 2.13. This finishes the control of ${}^*\sigma_{\mathcal{N}}$.

Control of $^*(\widehat{\phi})$. First we show that for $w \geq 2$,

$$\begin{aligned} \|{}^*(\widehat{\phi})_\psi\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}, \\ \|{}^*(\widehat{\phi})_\phi\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}. \end{aligned} \quad (4.102)$$

In (4.45), ${}^*(\widehat{\phi})$ was defined on each S_r , $r \geq 1$, as solution to

$$\mathcal{D}_2 \left({}^*(\widehat{\phi}) + \frac{1}{2} \nabla \widehat{\otimes} \epsilon^{[\geq 2]} \right) = \frac{1}{r^2} \nabla_N \left(r^2 \left(\frac{1}{2} \not{\phi}^{[\geq 2]} - {}^* \sigma_{\not{N}}^{[\geq 2]} - \frac{1}{2} \nabla \delta^{[\geq 2]} - \frac{1}{r} \epsilon^{[\geq 2]} \right) \right).$$

Using definitions (4.43) and (4.44), this can be decomposed into

$$\begin{aligned} \mathcal{D}_2 \left({}^*(\widehat{\phi})_\psi + \frac{1}{2} \nabla \widehat{\otimes} \epsilon_E^{[\geq 2]} \right) &= \frac{1}{r^2} \nabla_N \left(r^2 \left(-\zeta_E^{[\geq 2]} - \frac{1}{2} (\nabla \delta)_E^{[\geq 2]} - \frac{1}{r} \epsilon_E^{[\geq 2]} \right) \right), \\ \mathcal{D}_2 \left({}^*(\widehat{\phi})_\phi + \frac{1}{2} \nabla \widehat{\otimes} \epsilon_H^{[\geq 2]} \right) &= \frac{1}{r^2} \nabla_N \left(r^2 \left(\not{\phi}_H^{[\geq 2]} - \mathcal{D}_1^{-1} \left(0, \frac{1}{r^3} \partial_r \left(r^3 \sigma_{NN}^{[\geq 2]} \right) \right) - \frac{1}{r} \epsilon_H^{[\geq 2]} \right) \right). \end{aligned} \quad (4.103)$$

To analyse these equations, we first rewrite the second equation.

Claim 4.16. *The second equation of (4.103) is equivalent to*

$$\begin{aligned} \mathcal{D}_2 \left({}^*(\widehat{\phi})_\phi + \frac{1}{2} \nabla \widehat{\otimes} \epsilon_H^{[\geq 2]} \right) &= \frac{3}{2r} \not{\phi}_H^{[\geq 2]} + \frac{1}{r} \zeta_H^{[\geq 2]} - \frac{3}{r} {}^* \sigma_{\not{N}H}^{[\geq 2]} - \frac{1}{r^2} \nabla_N \left(r \epsilon_H^{[\geq 2]} \right) \\ &\quad + \mathcal{D}_1^{-1} \left(0, \not{\Delta} \sigma_{NN}^{[\geq 2]} \right) - \nabla_N \zeta_H^{[\geq 2]}. \end{aligned} \quad (4.104)$$

Proof. In the following, we use that for a scalar function $f^{[\geq 1]}$,

$$\nabla_N \left(\frac{1}{r} \mathcal{D}_1^{-1}(0, f) \right) = \frac{1}{r} \mathcal{D}_1^{-1}(0, \partial_r f^{[\geq 1]}),$$

this follows by Lemma 2.35.

By the definition of $\sigma_{NN}^{[\geq 2]}$ in (4.37) and of $^*\sigma_{NH}^{[\geq 2]}$ in (4.44), and using Lemma 2.35,

$$\begin{aligned}
& -\frac{1}{r^2} \nabla_N \left(r^2 \mathcal{P}_1^{-1} \left(0, \frac{1}{r^3} \partial_r \left(r^3 \sigma_{NN}^{[\geq 2]} \right) \right) \right) \\
&= -\frac{3}{r} \mathcal{P}_1^{-1} \left(0, \frac{1}{r^3} \partial_r \left(r^3 \sigma_{NN}^{[\geq 2]} \right) \right) - \mathcal{P}_1^{-1} \left(0, \partial_r \left(\frac{1}{r^3} \partial_r \left(r^3 \sigma_{NN}^{[\geq 2]} \right) \right) \right) \\
&= -\frac{3}{r} \left(\frac{1}{2} \rho_H^{[\geq 2]} + ^*\sigma_{NH}^{[\geq 2]} \right) - \mathcal{P}_1^{-1} \left(0, -\Delta \sigma_{NN}^{[\geq 2]} + \partial_r \text{curl} \left(\rho_H^{[\geq 2]} + \zeta_H^{[\geq 2]} \right) \right) \\
&= -\frac{3}{r} \left(\frac{1}{2} \rho_H^{[\geq 2]} + ^*\sigma_{NH}^{[\geq 2]} \right) + \mathcal{P}_1^{-1} \left(0, \Delta \sigma_{NN}^{[\geq 2]} \right) \\
&\quad - \mathcal{P}_1^{-1} \left(0, \partial_r \text{curl} \left(\rho_H^{[\geq 2]} + \zeta_H^{[\geq 2]} \right) \right) \\
&= -\frac{3}{r} \left(\frac{1}{2} \rho_H^{[\geq 2]} + ^*\sigma_{NH}^{[\geq 2]} \right) + \mathcal{P}_1^{-1} \left(0, \Delta \sigma_{NN}^{[\geq 2]} \right) \\
&\quad - \nabla_N \left(\rho_H^{[\geq 2]} + \zeta_H^{[\geq 2]} \right) + \frac{1}{r} \left(\rho_H^{[\geq 2]} + \zeta_H^{[\geq 2]} \right) \\
&= -\frac{3}{r} ^*\sigma_{NH}^{[\geq 2]} + \mathcal{P}_1^{-1} \left(0, \Delta \sigma_{NN}^{[\geq 2]} \right) - \nabla_N \left(\rho_H^{[\geq 2]} + \zeta_H^{[\geq 2]} \right) + \frac{1}{r} \left(-\frac{1}{2} \rho_H^{[\geq 2]} + \zeta_H^{[\geq 2]} \right).
\end{aligned}$$

Plugging this into the second equation of (4.103) finishes the proof of Claim 4.16. \square

By differentiating the first of (4.103) and (4.104), and using the commutation relations of Lemma 2.35, we can apply Propositions 2.23 and 2.24 to get for $w \geq 2$

$$\begin{aligned}
\| ^*(\widehat{\phi})_\psi \|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \| \epsilon \|_{\mathcal{H}_{-3/2}^{w-1}(\mathbb{R}^3 \setminus \overline{B_1})} + \| \zeta_E \|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3)} \\
&\quad + \| \mathcal{P}_2^{-1}(\nabla_N \zeta_E) \|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} + \| \delta \|_{\mathcal{H}_{-3/2}^{w-1}(\mathbb{R}^3 \setminus \overline{B_1})} \\
&\lesssim \| \rho \|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})},
\end{aligned} \tag{4.105}$$

$$\begin{aligned}
\| ^*(\widehat{\phi})_\phi \|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \| \epsilon \|_{\mathcal{H}_{-3/2}^{w-1}(\mathbb{R}^3 \setminus \overline{B_1})} + \| \mathcal{P}_2^{-1}(\nabla_N \zeta_H) \|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} \\
&\quad + \| \zeta_H \|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} + \| \rho_H^{[\geq 2]} \|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} \\
&\quad + \| \sigma_{NN}^{[\geq 2]} \|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} + \| ^*\sigma_{NH}^{[\geq 2]} \|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} \\
&\lesssim \| \rho \|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})},
\end{aligned} \tag{4.106}$$

where we used Proposition 4.11. This proves (4.102) for all $w \geq 2$.

It remains to show that $^*(\widehat{\phi}) \in \overline{\mathcal{H}}_{-5/2}^{w-2}$. This follows by (4.103), (4.104), by the fact that $\nabla_N \delta, \rho, \zeta_E, \zeta_H \in \overline{\mathcal{H}}_{-5/2}^{w-2}$, $\delta, \epsilon \in \overline{\mathcal{H}}_{-3/2}^{w-1}$ and $\mathcal{P}_2^{-1}(\nabla_N \zeta_E), \mathcal{P}_2^{-1}(\nabla_N \zeta_H) \in \overline{\mathcal{H}}_{-5/2}^{w-2}$ together with Proposition 2.13. Indeed, to show for a S_r -tangent symmetric 2-tensor $V_\psi^{[\geq 2]}$ that

$V|_{r=1} = 0$, it suffices to prove that $\mathrm{div}(V)|_{r=1} = 0$. Together with the commutation relations of Lemma 2.35, this concludes the control of $^*(\widehat{\phi})$.

Control of $\hat{\eta}$. First we prove for $w \geq 2$,

$$\|\hat{\eta}\|_{\mathcal{H}_{-3/2}^{w-1}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}. \quad (4.107)$$

The control of $\hat{\eta}$ follows by the control of the above quantities. Indeed, recall (4.46),

$$\mathcal{D}_2 \hat{\eta} = \frac{1}{2} \delta^{[\geq 2]} - {}^* \sigma_N^{[\geq 2]} - \frac{1}{2} \nabla \delta^{[\geq 2]} - \frac{1}{r} \epsilon^{[\geq 2]}. \quad (4.108)$$

Higher tangential regularity follows by Propositions 2.23 and 2.24. Indeed, tangentially deriving (4.108) yields for all $w \geq 2$, by the above control of $\delta, \epsilon, {}^* \sigma_N$,

$$\|\hat{\eta}\|_{\mathcal{H}_{-3/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} + \sum_{n=1}^{w-1} \|\nabla^n \hat{\eta}\|_{\mathcal{H}_{-3/2-n}^0(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}.$$

For radial regularity, use that by Lemma 4.10, $\hat{\eta}$ also solves (S2.5), that is,

$$\nabla_N \hat{\eta} + \frac{1}{r} \hat{\eta} = {}^*(\widehat{\phi}) + \frac{1}{2} \nabla \widehat{\epsilon}^{[\geq 2]}.$$

Differentiating in r yields with the above control of $\delta, \epsilon, {}^* \sigma_N, {}^*(\widehat{\phi})$ for all $w \geq 2$ the bound

$$\|\nabla_N^{w-1} \hat{\eta}\|_{\mathcal{H}_{-1/2-w}^0(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}.$$

This proves (4.107) for $w \geq 2$.

It remains to show that $\hat{\eta} \in \overline{\mathcal{H}}_{-3/2}^{w-1}$. This follows by (S2.5) and $\epsilon \in \overline{\mathcal{H}}_{-3/2}^{w-1}, {}^*(\widehat{\phi}) \in \overline{\mathcal{H}}_{-5/2}^{w-2}$ together with Proposition 2.13. This finishes the control of $\hat{\eta}$ as well as the proof of Lemma 4.15. \square

5. THE PRESCRIBED SCALAR CURVATURE EQUATION FOR g

In this section we prove the following theorem.

Theorem 5.1 (Metric extension theorem, version 2). *Let $w \geq 2$ be an integer. There exists a universal constant $\varepsilon > 0$ such that the following holds.*

- (1) **Extension result.** *Let $\bar{g} \in \mathcal{H}^w(B_1)$ be a Riemannian metric on the unit ball $B_1 \subset \mathbb{R}^3$ with scalar curvature $R(\bar{g})$, and let $R \in H_{-5/2}^{w-2}$ be such that $R|_{B_1} = R(\bar{g})$. If*

$$\|\bar{g} - e\|_{\mathcal{H}^w(B_1)} + \|R\|_{H_{-5/2}^{w-2}} < \varepsilon, \quad (5.1)$$

then there exists an $\mathcal{H}_{-1/2}^w$ -asymptotically flat metric \check{g} on \mathbb{R}^3 such that $\check{g}|_{B_1} = \bar{g}$ and its scalar curvature satisfies

$$R(\check{g}) = R \text{ on } \mathbb{R}^3,$$

Moreover, it is bounded by

$$\|\check{g} - e\|_{\mathcal{H}_{-1/2}^w} \lesssim \|\bar{g} - e\|_{\mathcal{H}^w(B_1)} + \|R\|_{H_{-5/2}^{w-2}}, \quad (5.2)$$

- (2) **Iteration estimates.** Let $\bar{g} \in \mathcal{H}^w(B_1)$ be a Riemannian metric on B_1 and let $R, \tilde{R} \in H_{-5/2}^{w-2}$ such that $R|_{B_1} = \tilde{R}|_{B_1} = R(\bar{g})$ and (5.1) holds for (\bar{g}, R) and (\bar{g}, \tilde{R}) . Let \check{g} and $\tilde{\check{g}}$ denote the metrics constructed in part (1) of this theorem with respect to R and \tilde{R} . Then

$$\|\check{g} - \tilde{\check{g}}\|_{\mathcal{H}_{-1/2}^w} \lesssim \|R - \tilde{R}\|_{H_{-5/2}^{w-2}}. \quad (5.3)$$

Before turning to the proof of Theorem 5.1, we first analyse in more detail the scalar curvature functional in the next section.

5.1. Scalar curvature and geometry of foliations. In this section, we analyse the scalar curvature functional with respect to the foliation of \mathbb{R}^3 by spheres S_r .

Lemma 5.2. Let g be a Riemannian metric on $\mathbb{R}^3 \setminus \overline{B_1}$,

$$g = a^2 dr + \gamma_{AB}(\beta^A dr + d\theta^A)(\beta^B dr + d\theta^B).$$

Then the scalar curvature $R(g)$ of g on $\mathbb{R}^3 \setminus \overline{B_1}$ is given by

$$R(g) = 2N(\text{tr}\Theta) - \frac{2}{a} \Delta_\gamma a + 2K(\gamma) - (\text{tr}\Theta)^2 - |\Theta|_\gamma^2,$$

where N denotes the unit normal and Θ the second fundamental form of $S_r \subset \mathbb{R}^3$ with respect to g , respectively, and $K(\gamma)$ the Gauss curvature of (S_r, γ) .

Proof. The lemma follows by the traced second variation equation⁶

$$N(\text{tr}\Theta) = \frac{1}{a} \Delta_\gamma a + \text{Ric}(N, N) + |\Theta|_\gamma^2$$

and the twice traced Gauss equation

$$R(g) = 2\text{Ric}(N, N) + 2K(\gamma) - (\text{tr}\Theta)^2 + |\Theta|_\gamma^2,$$

where Ric denotes the Ricci tensor of g . See Section 1 of [29] for a detailed derivation. This finishes the proof of Lemma 5.2. \square

We consider now variations of the metric and see how the scalar curvature changes.

Lemma 5.3. Let on $\mathbb{R}^3 \setminus \overline{B_1}$ be given a Riemannian metric g ,

$$g = a^2 dr + \gamma_{AB}(\beta^A dr + d\theta^A)(\beta^B dr + d\theta^B),$$

and a scalar function φ . Consider then the variation

$$\check{g}_\varphi := a^2 dr + e^{2\varphi} \gamma_{AB}(\beta^A dr + d\theta^A)(\beta^B dr + d\theta^B).$$

⁶Recall our sign convention $\Theta(X, Y) = -g(X, \nabla_Y N)$.

It holds that

$$\begin{aligned}
R(\check{g}_\varphi) = & -4N(N\varphi) - 2e^{-2\varphi}\check{\Delta}_\gamma\varphi + 2N(\text{tr}_\gamma\Theta) \\
& - \frac{2}{a}e^{-2\varphi}\check{\Delta}_\gamma a + 2e^{-2\varphi}K(\gamma) \\
& - 6(N\varphi)^2 + 6(N\varphi)\text{tr}_\gamma\Theta - (\text{tr}_\gamma\Theta)^2 - |\Theta|_\gamma^2,
\end{aligned} \tag{5.4}$$

where N denotes the unit normal to S_r and Θ the second fundamental form of S_r with respect to g .

Proof. By Lemma 5.2, it holds that

$$\begin{aligned}
R(\check{g}_\varphi) = & 2\check{N}(\text{tr}_{e^{2\varphi}\gamma}\check{\Theta}) - \frac{2}{a}\check{\Delta}_{e^{2\varphi}\gamma}a \\
& + 2K(e^{2\varphi}\gamma) - (\text{tr}_{e^{2\varphi}\gamma}\check{\Theta})^2 - |\check{\Theta}|_{e^{2\varphi}\gamma}^2,
\end{aligned} \tag{5.5}$$

where $\check{N}, \check{\Theta}$ are the unit normal and second fundamental form of S_r with respect to \check{g}_φ , respectively.

Note that in any coordinates on S_r , for $A, B = 1, 2$,

$$\begin{aligned}
\check{N} = N &= \frac{1}{a}\partial_r - \frac{1}{a}\beta, \\
\check{\Theta}_{AB} &= -\frac{1}{2a}\partial_r(e^{2\varphi}\gamma_{AB}) + \frac{1}{2a}(\check{\mathcal{L}}_\beta e^{2\varphi}\gamma)_{AB} \\
&= -(N\varphi)e^{2\varphi}\gamma_{AB} + e^{2\varphi}\Theta_{AB}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\text{tr}_{e^{2\varphi}\gamma}\check{\Theta} &= -2(N\varphi) + \text{tr}_\gamma\Theta, \\
|\check{\Theta}|_{e^{2\varphi}\gamma}^2 &= \gamma^{AC}\gamma^{BD}(-(N\varphi)\gamma_{AB} + \Theta_{AB})(-(N\varphi)\gamma_{CD} + \Theta_{CD}) \\
&= 2(N\varphi)^2 - 2(N\varphi)\text{tr}_\gamma\Theta + |\Theta|_\gamma^2.
\end{aligned} \tag{5.6}$$

Moreover, it holds in general that

$$\begin{aligned}
K(e^{2\varphi}\gamma) &= e^{-2\varphi}(K(\gamma) - \check{\Delta}_g\varphi), \\
\check{\Delta}_{e^{2\varphi}\gamma}a &= e^{-2\varphi}\check{\Delta}_\gamma a.
\end{aligned} \tag{5.7}$$

Plugging (5.7) and (5.6) into (5.5) yields

$$\begin{aligned}
R(\check{g}_\varphi) &= -4N(N\varphi) + 2N(\text{tr}_\gamma \Theta) - \frac{2}{a}e^{-2\varphi} \not\Delta_\gamma a \\
&\quad + 2e^{-2\varphi}(K(\gamma) - \not\Delta_\gamma \varphi) \\
&\quad - (4(N\varphi)^2 - 4(N\varphi)\text{tr}_\gamma \Theta + (\text{tr}_\gamma \Theta)^2) \\
&\quad - 2(N\varphi)^2 + 2(N\varphi)\text{tr}_\gamma \Theta - |\Theta|_\gamma^2 \\
&= 4N(N\varphi) - 2N(\text{tr}_\gamma \Theta) - \frac{2}{a}e^{-2\varphi} \not\Delta_\gamma a \\
&\quad + 2e^{-2\varphi}(K(\gamma) - \not\Delta_\gamma \varphi) \\
&\quad - 6(N\varphi)^2 + 6(N\varphi)\text{tr}_\gamma \Theta - (\text{tr}_\gamma \Theta)^2 - |\Theta|_\gamma^2.
\end{aligned}$$

This finishes the proof of Lemma 5.3. \square

The scalar curvature functional is a smooth mapping.

Lemma 5.4. *Let $w \geq 2$ be an integer. There exists an $\varepsilon > 0$ such that the scalar curvature functional*

$$R : g \mapsto R(g)$$

is a smooth mapping from $B_\varepsilon(e)$ to $H_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})$ where

$$B_\varepsilon(e) = \left\{ g : \|g - e\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \right\}.$$

Furthermore, at the Euclidean metric e , $R(e) = 0$.

Proof. The fact that $R(g)$ is a smooth mapping around e follows by similar considerations as in Lemma 4.2 and is left to the reader. \square

We calculate now the linearisation of the scalar curvature in φ and β at the Euclidean metric.

Lemma 5.5. *The linearisation of the scalar curvature $R(\check{g}_\varphi)$ in (φ, β) at the Euclidean metric is given by*

$$D_{\varphi, \beta} R(\check{g}_\varphi)|_{(\varphi=0, g=e)}(u, \xi) = -4\partial_r^2 u - 2\not\Delta_\gamma u - \frac{12}{r}\partial_r u - \frac{4}{r^2}u + \frac{2}{r^3}\partial_r(r^3 \text{div} \xi).$$

Proof. Calculate the variation δ of each term of (5.4), that is,

$$\begin{aligned}
R(\check{g}_\varphi) &= -4N(N\varphi) - 2e^{-2\varphi} \not\Delta_\gamma \varphi + 2N(\text{tr}_\gamma \Theta) \\
&\quad - \frac{2}{a}e^{-2\varphi} \not\Delta_\gamma a + 2e^{-2\varphi} K(\gamma) \\
&\quad - 6(N\varphi)^2 + 6(N\varphi)\text{tr}_\gamma \Theta - (\text{tr}_\gamma \Theta)^2 - |\Theta|_\gamma^2,
\end{aligned}$$

with $\delta\varphi = u, \delta\beta = \xi$. First,

$$\begin{aligned}\delta(-4N(N\varphi)) &= -4\partial_r^2 u, \\ \delta(-2e^{-2\varphi}\mathring{\Delta}_\gamma\varphi) &= -2\mathring{\Delta}_\gamma u\end{aligned}$$

Next,

$$\begin{aligned}\delta(2N(\text{tr}_\gamma\Theta)) &= \delta(2(\partial_r - \beta)(\text{tr}_\gamma\Theta)) \\ &= 2\partial_r(\delta\text{tr}_\gamma\Theta) - 2\underbrace{\xi(\text{tr}_\gamma\Theta)}_{=0} \\ &= 2\partial_r(\text{div}\xi),\end{aligned}$$

where we used that $\xi = \delta\beta$ is S_r -tangent and

$$(\text{tr}_\gamma\Theta)\Big|_{(\varphi=0, g=e)} = -2/r.$$

Moreover,

$$\begin{aligned}\delta\left(-\frac{2}{a}e^{-2\varphi}\mathring{\Delta}_\gamma a\right) &= 0, \\ \delta(2e^{-2\varphi}K(\gamma)) &= -4uK(\overset{\circ}{\gamma}) = -\frac{4}{r^2}u, \\ \delta(-6(N\varphi)^2) &= 0 \\ \delta(6(N\varphi)(\text{tr}_\gamma\Theta)) &= 6\left(-\frac{2}{r}\right)\partial_r u = -\frac{12}{r}\partial_r u, \\ \delta(-(\text{tr}_\gamma\Theta)^2) &= -2\left(-\frac{2}{r}\right)\delta(\text{div}(\beta)) = \frac{4}{r}\text{div}\xi.\end{aligned}$$

Finally,

$$\begin{aligned}\delta(-|\Theta|_\gamma^2) &= -\delta\left(-\frac{1}{2a^2}\gamma^{AC}\gamma^{BD}(\mathcal{L}_\beta\gamma)_{AB}\partial_r(\gamma_{CD})\right) \\ &= \frac{1}{2}\overset{\circ}{\gamma}^{AC}\overset{\circ}{\gamma}^{BD}\left(\mathcal{L}_\xi\overset{\circ}{\gamma}\right)_{AB}\frac{2}{r}\overset{\circ}{\gamma}_{CD} \\ &= \frac{2}{r}\text{div}\xi,\end{aligned}$$

where we used that

$$|\Theta|_\gamma^2 = \gamma^{AC}\gamma^{BD}\left(-\frac{1}{2a}\partial_r(\gamma_{AB}) + \frac{1}{2a}(\mathcal{L}_\beta\gamma)_{AB}\right)\left(-\frac{1}{2a}\partial_r(\gamma_{CD}) + \frac{1}{2a}(\mathcal{L}_\beta\gamma)_{CD}\right)$$

and in polar coordinates on the sphere, $\overset{\circ}{\gamma}_{CD} \sim r^2$.

To summarise, we have

$$D_{\varphi,\beta}R(\check{g}_\varphi)|_{(\varphi=0,g=e)}(u,\xi) = -4\partial_r^2 u - 2\Delta_{\check{\gamma}} u - \frac{12}{r}\partial_r u - \frac{4}{r^2}u + 2\left(\partial_r \operatorname{div}\xi + \frac{3}{r}\operatorname{div}\xi\right).$$

This finishes the proof of Lemma 5.5. \square

5.2. Reduction to the Euclidean case. In this section, we first prove the next perturbation result, Proposition 5.6, under the assumption of Lemma 5.8 which is proved in Section 5.3. Then we prove Theorem 5.1.

Proposition 5.6. *There is a small universal constant $\varepsilon > 0$ such that the following holds.*

- (1) **Extension result.** *Let g be an $\mathcal{H}_{-1/2}^w$ -asymptotically flat metric on $\mathbb{R}^3 \setminus \overline{B_1}$ written in standard polar coordinates as*

$$g = a^2 dr^2 + \gamma_{AB} (\beta^A dr + d\theta^A) (\beta^B dr + d\theta^B),$$

and $\mathfrak{s} \in \overline{H}_{-5/2}^{w-2}$ a scalar function. If

$$\|g - e\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} + \|\mathfrak{s}\|_{\overline{H}_{-5/2}^{w-2}} < \varepsilon, \quad (5.8)$$

then there exist a scalar function $\varphi \in \overline{H}_{-1/2}^w$ and an S_r -tangent vector $\beta' \in \overline{\mathcal{H}}_{-1/2}^w$ bounded by

$$\|(\varphi, \beta')\|_{\overline{H}_{-1/2}^w \times \overline{\mathcal{H}}_{-1/2}^w} \lesssim \|\mathfrak{s}\|_{\overline{H}_{-5/2}^{w-2}}, \quad (5.9)$$

and such that the metric

$$\check{g}_{\varphi,\beta'} := a^2 dr^2 + e^{2\varphi} \gamma_{AB} ((\beta + \beta')^A dr + d\theta^A) ((\beta + \beta')^B dr + d\theta^B) \quad (5.10)$$

is $\mathcal{H}_{-1/2}^w$ -asymptotically flat and its scalar curvature is on $\mathbb{R}^3 \setminus \overline{B_1}$

$$R(\check{g}_{\varphi,\beta'}) = R(g) + \mathfrak{s}.$$

Furthermore, it is bounded by

$$\|\check{g}_{\varphi,\beta'} - e\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|g - e\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} + \|\mathfrak{s}\|_{\overline{H}_{-5/2}^{w-2}}. \quad (5.11)$$

- (2) **Iteration estimates.** *Let g be an $\mathcal{H}_{-1/2}^w$ -asymptotically flat metric on $\mathbb{R}^3 \setminus \overline{B_1}$ written in standard polar coordinates as*

$$g = a^2 dr^2 + \gamma_{AB} (\beta^A dr + d\theta^A) (\beta^B dr + d\theta^B),$$

and $\mathfrak{s}, \tilde{\mathfrak{s}} \in \overline{H}_{-5/2}^{w-2}$ two scalar functions such that (5.8) holds for (g, \mathfrak{s}) and $(g, \tilde{\mathfrak{s}})$.

Applying part (1) to g with \mathfrak{s} and $\tilde{\mathfrak{s}}$ yields two pairs (φ, β') and $(\tilde{\varphi}, \tilde{\beta}')$, respectively. It holds that

$$\|(\varphi, \beta') - (\tilde{\varphi}, \tilde{\beta}')\|_{\overline{H}_{-1/2}^w \times \overline{\mathcal{H}}_{-1/2}^w} \lesssim \|\mathfrak{s} - \tilde{\mathfrak{s}}\|_{\overline{H}_{-5/2}^{w-2}}. \quad (5.12)$$

Let $\check{g}_{\varphi,\beta'}, \check{g}_{\tilde{\varphi},\tilde{\beta}'}$ denote the asymptotically flat metrics defined by (5.10). Then it holds that

$$\|\check{g}_{\varphi,\beta'} - \check{g}_{\tilde{\varphi},\tilde{\beta}'}\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\mathfrak{s} - \tilde{\mathfrak{s}}\|_{\overline{H}_{-5/2}^{w-2}}. \quad (5.13)$$

The proof of Proposition 5.6 is based on the Implicit Function Theorem and the essential Lemma 5.8 below which is proved in Section 5.3.

Definition 5.7. Let $\varphi \in \overline{H}_{-1/2}^w$ be a scalar function, $\beta' \in \overline{\mathcal{H}}_{-1/2}^w$ a S_r -tangent vectorfield and g a $\mathcal{H}_{-1/2}^w$ -asymptotically flat metric,

$$g = a^2 dr^2 + \gamma_{AB} (\beta^A dr + d\theta^A) (\beta^B dr + d\theta^B),$$

and let further

$$\check{g}_{\varphi,\beta'} := a^2 dr^2 + e^{2\varphi} \gamma_{AB} ((\beta + \beta')^A dr + d\theta^A) ((\beta + \beta')^B dr + d\theta^B).$$

Then, define

$$\mathcal{S}(\varphi, \beta', g) := R(\check{g}_{\varphi,\beta'}) - R(g).$$

It is left to the reader to verify that for (φ, β') close to $(0, 0)$ in $\overline{H}_{-1/2}^w \times \overline{\mathcal{H}}_{-1/2}^w$ and g close to e in $\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})$, the mapping \mathcal{S} is smooth and maps into $\overline{H}_{-5/2}^{w-2}$. Furthermore, it holds by construction that

$$\begin{aligned} & D_{\varphi,\beta'} \mathcal{S}|_{(0,0,e)}(u, \xi) \\ &= -4\partial_r^2 u - 2\Delta_{\check{\gamma}} u - \frac{12}{r} \partial_r u - \frac{4}{r^2} u + \frac{2}{r^3} \partial_r (r^3 \operatorname{div}_{\check{\gamma}} \xi), \end{aligned}$$

see Lemmas 5.4 and 5.5.

The next lemma is essential for the proof of Proposition 5.6.

Lemma 5.8 (Surjectivity at the Euclidean metric). *The linearization*

$$D_{\varphi,\beta'} \mathcal{S}|_{(0,0,e)} : \overline{H}_{-1/2}^w \times \overline{\mathcal{H}}_{-1/2}^w \rightarrow \overline{H}_{-5/2}^{w-2}$$

is surjective and has a bounded right-inverse. That is, for any $h \in \overline{H}_{-5/2}^{w-2}$, there exists $(u, \xi) \in \overline{H}_{-1/2}^w \times \overline{\mathcal{H}}_{-1/2}^w$ that solves

$$\begin{cases} 4\partial_r^2 u + \frac{12}{r} \partial_r u + \frac{4}{r^2} u + 2\Delta_{\check{\gamma}} u - \frac{2}{r^3} \partial_r (r^3 \operatorname{div}_{\check{\gamma}} \xi) = h & \text{on } \mathbb{R}^3 \setminus \overline{B_1}, \\ (u, \xi)|_{r=1} = 0. \end{cases}$$

Furthermore, it is bounded by

$$\|(u, \xi)\|_{\overline{H}_{-1/2}^w \times \overline{\mathcal{H}}_{-1/2}^w} \lesssim \|h\|_{\overline{H}_{-5/2}^{w-2}}.$$

Let

$$\overline{\mathcal{N}}_e := \ker \left(D_{\varphi, \beta'} \mathcal{S}|_{(0,0,e)} \right)^\perp \subset \overline{H}_{-1/2}^w \times \overline{\mathcal{H}}_{-1/2}^w,$$

where \perp denotes the orthogonal complement with respect to the scalar product on $\overline{H}_{-1/2}^w \times \overline{\mathcal{H}}_{-1/2}^w$. $\overline{\mathcal{N}}_e$ is a closed subspace of $\overline{H}_{-1/2}^w \times \overline{\mathcal{H}}_{-1/2}^w$ and therefore also Hilbert. From now on, let \mathcal{S} be restricted to $(\varphi, \beta') \in \overline{\mathcal{N}}_e$.

We are now ready to prove Proposition 5.6.

Proof of Proposition 5.6. By Lemma 5.8, the linearization $D_{\varphi, \beta'} \mathcal{S}|_{(0,0,e)}$ is an isomorphism, and $\mathcal{S}(0,0,e) = 0$. Therefore, by the Inverse Function Theorem 2.37, there are open neighbourhoods V_0 around the Euclidean metric e and $W_0 \subset \overline{H}_{-5/2}^{w-2}$ around 0, together with a unique smooth mapping

$$\begin{aligned} \mathcal{G} : V_0 \times W_0 &\rightarrow \overline{H}_{-1/2}^w \times \overline{\mathcal{H}}_{-1/2}^w, \\ (g, \mathfrak{s}) &\mapsto \mathcal{G}(g, \mathfrak{s}) := (\varphi, \beta') \end{aligned}$$

such that for $g \in V_0, \mathfrak{s} \in W_0$, on $\mathbb{R}^3 \setminus \overline{B_1}$,

$$\mathcal{S}(\mathcal{G}(g, \mathfrak{s}), g) = \mathfrak{s}.$$

Moreover, it holds by the uniqueness of \mathcal{G} that for every $g \in V_0$,

$$\mathcal{G}(g, 0) = 0, \tag{5.14}$$

because $\mathcal{S}(0,0,g) = R(g) - R(g) = 0$.

There exists $\varepsilon > 0$ small such that for (g, \mathfrak{s}) with

$$\|g - e\|_{\mathcal{H}_{-1/2}^w} < \varepsilon, \|\mathfrak{s}\|_{\overline{H}_{-5/2}^{w-2}} < \varepsilon$$

it holds that $g \in V_0, \mathfrak{s} \in W_0$ and furthermore, for

$$(\varphi, \beta') := \mathcal{G}(\tilde{g}, \mathfrak{s})$$

it holds that

$$\begin{aligned} \|(u, \xi)\|_{\overline{H}_{-1/2}^w \times \overline{\mathcal{H}}_{-1/2}^w} &= \|\mathcal{G}(\tilde{g}, \mathfrak{s})\|_{\overline{H}_{-1/2}^w \times \overline{\mathcal{H}}_{-1/2}^w} \\ &= \|\mathcal{G}(\tilde{g}, \mathfrak{s}) - \underbrace{\mathcal{G}(\tilde{g}, 0)}_{=0}\|_{\overline{H}_{-1/2}^w \times \overline{\mathcal{H}}_{-1/2}^w} \\ &\lesssim \|\mathfrak{s}\|_{\overline{H}_{5/2}^{w-2}}, \end{aligned} \tag{5.15}$$

see Lemma 2.38. This proves (5.9).

We now prove (5.12). By Lemma 2.38, there is a $\varepsilon > 0$ small such that for $g, \mathfrak{s}, \tilde{\mathfrak{s}}$ with

$$\|g - e\|_{\mathcal{H}_{-1/2}^w} < \varepsilon, \|\mathfrak{s}\|_{\overline{H}_{-5/2}^{w-2}} < \varepsilon, \|\tilde{\mathfrak{s}}\|_{\overline{H}_{-5/2}^{w-2}} < \varepsilon$$

it holds that

$$\begin{aligned} \|(\varphi, \beta') - (\tilde{\varphi}, \tilde{\beta}')\|_{\overline{H}_{-1/2}^w \times \overline{H}_{-1/2}^w} &= \|\mathcal{G}(g, \mathfrak{s}) - \mathcal{G}(g, \tilde{\mathfrak{s}})\|_{\overline{H}_{-1/2}^w \times \overline{H}_{-1/2}^w} \\ &\lesssim \|\mathfrak{s} - \tilde{\mathfrak{s}}\|_{\overline{H}_{-5/2}^{w-2}}. \end{aligned} \quad (5.16)$$

This proves (5.12).

Estimates for g . To prove (5.11) for $\varepsilon > 0$ small, it suffices by Part (2) of Lemma 2.22 to prove

$$\|a^2 - 1\|_{\mathcal{H}_{-1/2}^w} + \|\beta^A + \beta'^A\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} + \|e^{2\varphi}\gamma - \overset{\circ}{\gamma}\|_{H_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|g - e\|_{\mathcal{H}_{-1/2}^w} + \|\mathfrak{s}\|_{\overline{H}_{-5/2}^{w-2}}.$$

First, the control of a^2 is immediate because a^2 remained the same in the variation. Therefore, by Part (1) of Lemma 2.22, for $\varepsilon > 0$ sufficiently small,

$$\|a^2 - 1\|_{\mathcal{H}_{-1/2}^w} \lesssim \|g - e\|_{\mathcal{H}_{-1/2}^w}.$$

Second, by (5.9) and Lemma 2.22,

$$\begin{aligned} \|\beta + \beta'\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|\beta\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} + \|\beta'\|_{\overline{H}_{-1/2}^w} \\ &\lesssim \|\beta\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} + \|\mathfrak{s}\|_{\overline{H}_{-5/2}^{w-2}} \\ &\lesssim \|g - e\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} + \|\mathfrak{s}\|_{\overline{H}_{-5/2}^{w-2}}. \end{aligned}$$

Third, we claim that for $\varepsilon > 0$ small

$$\|e^{2\varphi}\gamma - \overset{\circ}{\gamma}\|_{H_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|g - e\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} + \|\mathfrak{s}\|_{\overline{H}_{-5/2}^{w-2}}.$$

Indeed, estimate

$$\|e^{2\varphi}\gamma - \overset{\circ}{\gamma}\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \leq \left\| (e^{2\varphi} - 1) \overset{\circ}{\gamma} \right\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} + \left\| e^{2\varphi} (\gamma - \overset{\circ}{\gamma}) \right\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})}. \quad (5.17)$$

The first term on the right-hand side of (5.17) can be bounded by Corollary 2.11 and (5.9) for $\varepsilon > 0$ small enough,

$$\begin{aligned} \left\| (e^{2\varphi} - 1) \overset{\circ}{\gamma} \right\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|\varphi\|_{\overline{H}_{-1/2}^w} \\ &\lesssim \|\mathfrak{s}\|_{\overline{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}. \end{aligned}$$

The second term on the right-hand side of (5.17) is estimated by Lemmas 2.9 and 2.22 with (5.9). Namely, for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned} \left\| e^{2\varphi} (\gamma - \overset{\circ}{\gamma}) \right\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|\gamma - \overset{\circ}{\gamma}\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \\ &\lesssim \|g - e\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})}. \end{aligned}$$

Together, this shows that

$$\|e^{2\varphi}\gamma - \overset{\circ}{\gamma}\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|g - e\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} + \|\mathfrak{s}\|_{\overline{H}_{-5/2}^{w-2}}.$$

This finishes the proof of (5.11).

It remains to prove the iteration estimate (5.13). Indeed, by the construction of $\check{g}_{\varphi, \beta'}$, $\check{g}_{\tilde{\varphi}, \tilde{\beta}'}$ as variations of g with (φ, β') , $(\tilde{\varphi}, \tilde{\beta}')$ we can write, see Section 2.2, with $B = 1, 2$,

$$\begin{aligned} (\check{g}_{\varphi, \beta'} - \check{g}_{\tilde{\varphi}, \tilde{\beta}'})_{NN} &= a^2 - a^2 = 0, \\ \check{g}_{\varphi, \beta'} - \check{g}_{\tilde{\varphi}, \tilde{\beta}'} &= (e^{2\varphi} - e^{2\tilde{\varphi}})\gamma, \\ \left(\check{g}_{\varphi, \beta'_N} - \check{g}_{\tilde{\varphi}, \tilde{\beta}'_N}\right)_B &= e^{2\varphi}\gamma_{BA}(\beta^A + \beta'^A) - e^{2\tilde{\varphi}}\gamma_{BA}(\beta^A + \tilde{\beta}'^A) \\ &= (e^{2\varphi} - e^{2\tilde{\varphi}})\gamma_{BA}(\beta^A + \beta'^A) + e^{2\tilde{\varphi}}\gamma_{BA}(\beta'^A - \tilde{\beta}'^A). \end{aligned} \tag{5.18}$$

By Corollary 2.11 and (5.12) it holds for $\varepsilon > 0$ sufficiently small that

$$\|e^{2\varphi} - e^{2\tilde{\varphi}}\|_{\overline{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\mathfrak{s} - \tilde{\mathfrak{s}}\|_{\overline{H}_{-5/2}^{w-2}}.$$

For $\varepsilon > 0$ small enough, we can use this estimate and the expression (5.18) with (5.12) to apply Lemmas 2.9, 2.22 and 2.19 to get

$$\|\check{g}_{\varphi, \beta'} - \check{g}_{\tilde{\varphi}, \tilde{\beta}'}\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\mathfrak{s} - \tilde{\mathfrak{s}}\|_{\overline{H}_{-5/2}^{w-2}}.$$

This proves (5.13) and finishes the proof of Proposition 5.6. \square

We are now in position to prove Theorem 5.1.

Proof of Theorem 5.1. We prove the two parts of the theorem separately.

Proof of Part (1). Use standard Sobolev extension to extend \bar{g} from B_1 to an asymptotically flat metric g on \mathbb{R}^3 such that

$$\|g - e\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3)} \lesssim \|\bar{g} - e\|_{\mathcal{H}^w(B_1)}. \tag{5.19}$$

Denote its standard polar components on $\mathbb{R}^3 \setminus \overline{B_1}$ by

$$g = a^2 dr^2 + \gamma_{AB}(\beta^A dr + e^A)(\beta^B dr + e^B).$$

Given a $R \in H_{-5/2}^{w-2}$ such that $R|_{B_1} = R(\bar{g})$, let

$$\mathfrak{s} := R - R(g) \in \overline{H}_{-5/2}^{w-2},$$

where we used Proposition 2.13. It holds for $\varepsilon > 0$ small that

$$\begin{aligned} \|\mathfrak{s}\|_{\overline{H}_{-5/2}^{w-2}} &\leq \|R\|_{H_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} + \|R(g)\|_{H_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})}, \\ &\lesssim \|R\|_{H_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} + \|g - e\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})}. \end{aligned} \tag{5.20}$$

Therefore, for $\varepsilon > 0$ small enough, Proposition 5.6 yields a pair (φ, β') such that

$$\check{g} = a^2 dr^2 + e^{2\varphi} \gamma_{AB} ((\beta + \beta')^A dr + e^A) ((\beta + \beta')^B dr + e^B)$$

is a $\mathcal{H}_{-1/2}^w$ -asymptotically flat metric with $\check{g}|_{B_1} = \bar{g}$ and scalar curvature

$$R(\check{g}) = R(g) + S = R.$$

By (5.11), (5.19) and (5.20),

$$\begin{aligned} \|\check{g} - e\|_{\mathcal{H}_{-1/2}^w} &\lesssim \|g - e\|_{\mathcal{H}_{-1/2}^w} + \|\mathfrak{s}\|_{\overline{H}_{-5/2}^{w-2}}, \\ &\lesssim \|\bar{g} - e\|_{\mathcal{H}^w(B_1)} + \|R\|_{H_{-5/2}^{w-2}}. \end{aligned}$$

This proves (5.2). This finishes the proof of part (1) of Theorem 5.1.

Proof of Part (2). Use standard Sobolev extension to extend \bar{g} from B_1 to an asymptotically flat metric g on \mathbb{R}^3 such that

$$\|g - e\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3)} \lesssim \|\bar{g} - e\|_{\mathcal{H}^w(B_1)}.$$

Let $\mathfrak{s} := R - R(g)$, $\tilde{\mathfrak{s}} := \tilde{R} - R(g)$, so that

$$\mathfrak{s} - \tilde{\mathfrak{s}} = (R(g) + \mathfrak{s}) - (R(g) + \tilde{\mathfrak{s}}) = R - \tilde{R}.$$

Hence, for $\varepsilon > 0$ sufficiently small, (5.3) follows from (5.13) in Proposition 5.6. This finishes the proof of Theorem 5.1. \square

5.3. Surjectivity at the Euclidean metric. In this section, we prove Lemma 5.8, that is, we show that for every scalar function $h \in \overline{H}_{-5/2}^{w-2}$ there exist $(u, \xi) \in \overline{H}_{-1/2}^w \times \overline{\mathcal{H}}_{-1/2}^w$ that solves on $\mathbb{R}^3 \setminus \overline{B_1}$

$$\partial_r^2 u + \frac{3}{r} \partial_r u + \frac{1}{r^2} u + \frac{1}{2} \Delta_{\check{g}} u - \frac{1}{2r^3} \partial_r \left(r^3 \operatorname{div}_{\check{g}} \xi \right) = \frac{1}{4} h \quad (5.21)$$

and is bounded by

$$\|(u, \xi)\|_{\overline{H}_{-1/2}^w \times \overline{\mathcal{H}}_{-1/2}^w} \lesssim \|h\|_{\overline{H}_{-5/2}^{w-2}}.$$

In this section all operators are Euclidean. We consider the following more general system on $\mathbb{R}^3 \setminus \overline{B_1}$

$$\begin{aligned} \Delta u + \frac{1}{r} \partial_r u + \frac{1}{r^2} u - \frac{1}{2} \Delta u &= \frac{1}{2} \left(\frac{1}{2} h + \zeta^{[\geq 1]} \right), \\ \frac{1}{r^3} \partial_r \left(r^3 \operatorname{div} \xi \right) &= \zeta^{[\geq 1]}, \end{aligned} \quad (5.22)$$

where $\zeta^{[\geq 1]} \in \overline{H}_{-5/2}^{w-2}$ is a scalar function. By the relation $\Delta = \partial_r^2 + \frac{2}{r} \partial_r + \frac{1}{r^2} \Delta$, it follows that for any $\zeta^{[\geq 1]}$, a solution (u, ξ) to (5.22) also solves (5.21). From now on, we will thus focus on (5.22).

5.3.1. *Construction of the solution (u, ξ) and $\zeta^{[\geq 1]}$.* In this section we construct two scalar functions $\zeta^{[\geq 1]}$, u and a S_r -tangent vectorfield ξ . It is shown in Section 5.3.2 that they are a solution to (5.22).

Let the given $h \in \overline{H}_{-5/2}^{w-2}$ be decomposed as

$$h = h^{[0]} + h^{[\geq 1]}.$$

• **Definition of u and $\zeta^{[\geq 1]}$.** Let the scalar function

$$u = u^{[0]} + u^{[\geq 1]},$$

where the radial function $u^{[0]}(r)$ is defined as solution to the following ODE on $r > 1$,

$$\begin{cases} \partial_r^2 u^{[0]} + \frac{3}{r} \partial_r u^{[0]} + \frac{1}{r^2} u^{[0]} = \frac{1}{4} h^{[0]}, \\ u^{[0]}|_{r=1} = \partial_r u^{[0]}|_{r=1} = 0, \end{cases} \quad (5.23)$$

and $u^{[\geq 1]}$ is defined as solution to the elliptic PDE on $\mathbb{R}^3 \setminus \overline{B_1}$

$$\begin{cases} \Delta u^{[\geq 1]} - \frac{1}{2} \not\Delta u^{[\geq 1]} + \frac{1}{r} \partial_r u^{[\geq 1]} + \frac{1}{r^2} u^{[\geq 1]} = \frac{1}{2} \left(\frac{1}{2} h^{[\geq 1]} + \zeta^{[\geq 1]} \right), \\ u^{[\geq 1]}|_{r=1} = 0, \end{cases} \quad (5.24)$$

where on \mathbb{R}^3 ,

$$\begin{aligned} \zeta^{[\geq 1]} &:= \sum_{l \geq 1} \sum_{m=-l}^l \zeta^{(lm)} Y^{(lm)}, \\ \zeta^{(lm)} &:= c^{(lm)} r^{\sqrt{l(l+1)/2}-1} \partial_r (\chi(l(r-1))) \\ &\quad - \tilde{c}^{(lm)} r^{\sqrt{l(l+1)/2}-1} \partial_r^2 (\chi(l(r-1))), \end{aligned} \quad (5.25)$$

where χ denotes the smooth transition function defined in (2.1) and for $l \geq 1$,

$$\begin{aligned} c^{(lm)} &:= -\frac{1}{2} \int_1^\infty r^{1-\sqrt{l(l+1)/2}} h^{(lm)} dr, \\ \tilde{c}^{(lm)} &:= \frac{\int_1^\infty c^{(lm)} r^{\sqrt{l(l+1)/2}+1} \partial_r (\chi(l(r-1))) dr}{\int_1^\infty r^{\sqrt{l(l+1)/2}+1} \partial_r^2 (\chi(l(r-1))) dr}. \end{aligned} \quad (5.26)$$

- **Definition of ξ .** Let ξ be the S_r -tangent vectorfield solving on $\mathbb{R}^3 \setminus \overline{B_1}$

$$\begin{cases} \frac{1}{r^3} \partial_r (r^3 \operatorname{div} \xi) = \zeta^{[\geq 1]}, \\ \operatorname{curl} \xi = 0, \\ \xi|_{r=1} = 0. \end{cases} \quad (5.27)$$

5.3.2. *Proof of surjectivity at the Euclidean metric.* In this section, we prove first in Lemma 5.9 that $\zeta^{[\geq 1]}, u$ and β solve (5.22). Then, in Propositions 5.10 and 5.13, we show that

$$\zeta^{[\geq 1]} \in \overline{H}_{-5/2}^{w-2}, \quad (u, \xi) \in \overline{H}_{-1/2}^w \times \overline{\mathcal{H}}_{-1/2}^w$$

with quantitative bounds. These results prove Lemma 5.8.

Lemma 5.9. *The $u, \xi, \zeta^{[\geq 1]}$ defined in (5.23)-(5.27) solve (5.22), that is, on $\mathbb{R}^3 \setminus \overline{B_1}$,*

$$\begin{aligned} \Delta u + \frac{1}{r} \partial_r u + \frac{1}{r^2} u - \frac{1}{2} \Delta u &= \frac{1}{2} \left(\frac{1}{2} h + \zeta^{[\geq 1]} \right), \\ \frac{1}{r^3} \partial_r (r^3 \operatorname{div} \xi) &= \zeta^{[\geq 1]}. \end{aligned}$$

Proof of Lemma 5.9. The coefficients of the system (5.22) depend only on r . Therefore we may project the equations of (5.22) onto the Hodge-Fourier basis elements. This uses Proposition 2.30. We split (5.22) into the modes $l = 0$ and $l \geq 1$ and get the following two subsystems **S0**, **S1**.

$$\partial_r^2 u^{[0]} + \frac{3}{r} \partial_r u^{[0]} + \frac{1}{r^2} u^{[0]} = \frac{1}{4} h^{[0]}, \quad (\mathbf{S0})$$

$$\Delta u^{[\geq 1]} + \frac{1}{r} \partial_r u^{[\geq 1]} + \frac{1}{r^2} u^{[\geq 1]} - \frac{1}{2} \Delta u^{[\geq 1]} = \frac{1}{2} \left(\frac{1}{2} h^{[\geq 1]} + \zeta^{[\geq 1]} \right), \quad (\mathbf{S1.1})$$

$$\frac{1}{r^3} \partial_r (r^3 \operatorname{div} \xi) = \zeta^{[\geq 1]}. \quad (\mathbf{S1.2})$$

The (u, ξ) and $\zeta^{[\geq 1]}$ defined in (5.23)-(5.27) directly solve these equations on $\mathbb{R}^3 \setminus \overline{B_1}$. This proves Lemma 5.9. \square

The next proposition is essential and only possible due to our careful choice of $\zeta^{[\geq 1]}$ which ensures that for all $w \geq 2$, we have the boundary behaviour $u^{[\geq 1]} \in \overline{H}_{-1/2}^w$.

Proposition 5.10. *Let $w \geq 2$ be an integer. The following holds.*

- **Regularity and boundary control of $\zeta^{[\geq 1]}$.** *The scalar function $\zeta^{[\geq 1]}$ is well-defined by (5.25)-(5.26) and bounded by*

$$\|\zeta^{[\geq 1]}\|_{H_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|h^{[\geq 1]}\|_{\overline{H}_{-5/2}^{w-2}}. \quad (5.28)$$

Moreover, $\zeta^{[\geq 1]} \in \overline{H}_{-5/2}^{w-2}$ and the following integral identities hold.

$$\int_1^\infty r^{1-\sqrt{l(l+1)/2}} \left(\frac{1}{2} h^{(lm)} + \zeta^{(lm)} \right) dr = 0, \quad (5.29)$$

$$\int_1^\infty r^2 \zeta^{(lm)} dr = 0. \quad (5.30)$$

- **Precise estimate for $\zeta^{[\geq 1]}$.** It holds that

$$\|\mathcal{P}_1^{-1}(\partial_r \zeta^{[\geq 1]}, 0)\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|h\|_{\overline{H}_{-5/2}^{w-2}}. \quad (5.31)$$

Moreover, $\mathcal{P}_1^{-1}(\partial_r \zeta^{[\geq 1]}, 0) \in \overline{\mathcal{H}}_{-5/2}^{w-2}$.

- **Regularity and boundary control of $u^{[\geq 1]}$.** The scalar function $u^{[\geq 1]}$ defined in (5.24) is bounded by

$$\|u^{[\geq 1]}\|_{H_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|h^{[\geq 1]}\|_{\overline{H}_{-5/2}^{w-2}}. \quad (5.32)$$

Moreover, $u^{[\geq 1]} \in \overline{H}_{-1/2}^w$ and in particular,

$$\partial_r u^{[\geq 1]}|_{r=1} = 0.$$

Remark 5.11. The function $u^{[\geq 1]}$ satisfies the elliptic equation (5.24). Therefore its boundary behaviour is harder to estimate than for $u^{[0]}, \xi$ which satisfy transport equations in r .

Proof. We prove each point separately.

Regularity and boundary control of $\zeta^{[\geq 1]}$. We show at first that the constants $c^{(lm)}, \tilde{c}^{(lm)}$ are well-defined in (5.26). Concerning $c^{(lm)}$, for $l \geq 1, m \in \{-l, \dots, l\}$,

$$\begin{aligned} |c^{(lm)}| &= \frac{1}{2} \left| \int_1^\infty r^{1-\sqrt{l(l+1)/2}} h^{(lm)} dr \right| \\ &\leq \frac{1}{2} \left(\int_1^\infty r^{-2\sqrt{l(l+1)/2}} dr \right)^{1/2} \left(\int_1^\infty (r h^{(lm)})^2 dr \right)^{1/2} \\ &= \frac{1}{2} \frac{1}{(2\sqrt{l(l+1)/2} - 1)^{1/2}} \left(\int_1^\infty (r h^{(lm)})^2 dr \right)^{1/2}. \end{aligned} \quad (5.33)$$

To estimate $\tilde{c}^{(lm)}$, estimate first its denominator. Integrating by parts twice and using that $\text{supp}(\partial_r \chi)(l(r-1)) \subset [1, 1 + \frac{1}{l}]$ yields

$$\begin{aligned}
& \int_1^\infty r^{\sqrt{l(l+1)/2}+1} \partial_r^2 (\chi(l(r-1))) dr \\
&= - \left(\sqrt{l(l+1)/2} + 1 \right) \int_1^{1+\frac{1}{l}} r^{\sqrt{l(l+1)/2}} \partial_r (\chi(l(r-1))) dr \\
&= - \left(\sqrt{l(l+1)/2} + 1 \right) \left(1 + \frac{1}{l} \right)^{\sqrt{l(l+1)/2}} \\
&\quad + \left(\sqrt{l(l+1)/2} + 1 \right) \left(\sqrt{l(l+1)/2} \right) \int_1^{1+\frac{1}{l}} r^{\sqrt{l(l+1)/2}-1} \chi(l(r-1)) dr \\
&\leq - \left(\sqrt{l(l+1)/2} + 1 \right),
\end{aligned}$$

where we uniformly bounded $|\chi| \leq 1$. Consequently, for all $l \geq 1$,

$$\begin{aligned}
|\tilde{c}^{(lm)}| &\leq \frac{1}{\sqrt{l(l+1)/2} + 1} \left| \int_1^\infty c^{(lm)} r^{\sqrt{l(l+1)/2}+1} \partial_r (\chi(l(r-1))) dr \right| \\
&\leq \frac{|c^{(lm)}| l}{\sqrt{l(l+1)/2} + 1} \int_1^{1+\frac{1}{l}} \left(1 + \frac{1}{l} \right)^{\sqrt{l(l+1)/2}+1} |\partial_r \chi|(l(r-1)) dr \\
&\lesssim \frac{|c^{(lm)}|}{l} \\
&\lesssim \frac{1}{l(2\sqrt{l(l+1)/2} + 1)^{1/2}} \left(\int_1^\infty (r h^{(lm)})^2 dr \right)^{1/2}
\end{aligned} \tag{5.34}$$

where we used (5.33) and the fact that $|\partial_r \chi|$ is universally bounded. This shows that $c^{(lm)}$, $\tilde{c}^{(lm)}$ are well-defined.

We now prove the case $w = 2$ of (5.28), that is,

$$\|\zeta^{[\geq 1]}\|_{H_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|h^{[\geq 1]}\|_{\overline{H}_{-5/2}^0}.$$

Indeed, by plugging in (5.25), for $l \geq 1$,

$$\begin{aligned}
& \int_1^\infty r^2 (\zeta^{(lm)})^2 dr \\
& \lesssim \int_1^\infty r^{2\sqrt{l(l+1)/2}} \left[(c^{(lm)})^2 (\partial_r (\chi(l(r-1))))^2 + (\tilde{c}^{(lm)})^2 (\partial_r^2 (\chi(l(r-1))))^2 \right] dr \\
& \lesssim \int_1^\infty r^{2\sqrt{l(l+1)/2}} \left[(c^{(lm)})^2 l^2 (\partial_r \chi)^2(l(r-1)) + (\tilde{c}^{(lm)})^2 l^4 (\partial_r^2 \chi)^2(l(r-1)) \right] dr \\
& \lesssim \left(\int_1^\infty (rh^{(lm)})^2 dr \right) l^{1+\frac{1}{l}} \int_1^\infty r^{2\sqrt{l(l+1)/2}} dr \\
& \lesssim \left(1 + \frac{1}{l} \right)^{2\sqrt{l(l+1)/2}} \int_1^\infty (rh^{(lm)})^2 dr \\
& \lesssim \int_1^\infty (rh^{(lm)})^2 dr,
\end{aligned} \tag{5.35}$$

where we used that $\text{supp } \partial_r \chi(l(r-1)) \subset [1, 1 + \frac{1}{l}]$ and (5.33), (5.34). Summing over $l \geq 1, m \in \{-l, \dots, l\}$ proves the case $w = 2$ of (5.28).

We turn now to the case $w > 2$ of (5.28). On the one hand, the estimates (5.33), (5.34) improve,

$$\begin{aligned}
|c^{(lm)}| & \lesssim \frac{1}{(2\sqrt{l(l+1)/2} - 1)^{1/2}} \frac{1}{\sqrt{l(l+1)}^{w-2}} \left(\int_1^\infty \left(\frac{l(l+1)}{r^2} \right)^{w-2} (r^{w-1} h^{(lm)})^2 dr \right)^{1/2}, \\
|\tilde{c}^{(lm)}| & \lesssim \frac{1}{l(2\sqrt{l(l+1)/2} + 1)^{1/2}} \frac{1}{\sqrt{l(l+1)}^{w-2}} \left(\int_1^\infty \left(\frac{l(l+1)}{r^2} \right)^{w-2} (r^{w-1} h^{(lm)})^2 dr \right)^{1/2},
\end{aligned} \tag{5.36}$$

where the integrals on the right-hand side correspond to the norm $\|h\|_{\overline{H}_{-5/2}^{w-2}}$ and are therefore summable, see Proposition 2.34.

On the other hand, by differentiating (5.25) and using Lemma 2.33, derivatives of $\zeta^{(lm)}$ generally are of the form

$$\partial_r \zeta^{(lm)} \approx \frac{l}{r} \zeta^{(lm)}, \quad (\nabla \zeta)_E^{(lm)} = -\frac{\sqrt{l(l+1)}}{r} \zeta^{(lm)}. \quad (5.37)$$

Combining (5.36) with (5.37), yields estimates for higher derivatives of $\zeta^{[\geq 1]}$ analogously to (5.35). This proves (5.28) for all $w \geq 2$, see Lemma 2.35.

Next, we claim that $\zeta^{[\geq 1]} \in \overline{H}_{-5/2}^{w-2}$. Indeed, the sequence of smooth functions

$$f_n := \sum_{l=1}^n \sum_{m=-l}^l \zeta^{(lm)} Y^{(lm)}$$

is compactly supported in $\mathbb{R}^3 \setminus \overline{B_1}$ and converges by (5.28) in $\overline{H}_{-5/2}^{w-2}$ to $\zeta^{[\geq 1]}$ as $n \rightarrow \infty$. See also the analogous (4.66), (4.67).

The integral identities (5.29) and (5.30) follow from the definition of $\zeta^{[\geq 1]}$ in (5.25)-(5.26). The proof is left to the reader, see the analogous identity (4.68).

Precise estimate for $\zeta^{[\geq 1]}$. The proof is similar to part (2) of Proposition 4.11 and therefore only sketched here.

Consider first the case $w = 2$ of (5.31). By Proposition 2.34 and Lemmas 2.35 and 2.36 and given that we already control $\zeta^{[\geq 1]}$ above, it suffices to prove in the Hodge-Fourier formalism that for $l \geq 1, m \in \{-l, \dots, l\}$,

$$\int_1^\infty r^2 \left(\frac{r}{\sqrt{l(l+1)}} \partial_r \zeta^{(lm)} \right)^2 dr \lesssim \int_1^\infty r^2 (h^{(lm)})^2 dr. \quad (5.38)$$

This follows by using the explicit (5.25) which shows that schematically

$$\begin{aligned} \left| \frac{r}{\sqrt{l(l+1)}} \partial_r \zeta^{(lm)} \right| &\lesssim \zeta^{(lm)} + c^{(lm)} r \sqrt{l(l+1)/2} \partial_r ((\partial_r \chi)(l(r-1))) \\ &\quad - \tilde{c}^{(lm)} r \sqrt{l(l+1)/2} \partial_r^2 ((\partial_r \chi)(l(r-1))) \\ &\approx \zeta^{(lm)}. \end{aligned}$$

Therefore, by the above control of $\zeta^{[\geq 1]}$, (5.38) follows. This proves (5.31) in the case $w = 2$.

The case $w > 2$ of (5.31) is treated similarly. Indeed, by the explicit expression (5.25), derivatives can be expressed in the Hodge-Fourier formalism as multiplication by $\frac{\sqrt{l(l+1)}}{r}$.

At the same time, the estimates for the constants $c^{(lm)}, \tilde{c}^{(lm)}$ improve, see (5.36). This allows to use the estimate above to conclude (5.31) for all $w \geq 2$.

It remains to show that $\mathcal{P}_1^{-1}(\partial_r \zeta^{[\geq 1]}, 0) \in \overline{\mathcal{H}}_{-5/2}^{w-2}$. By using (5.31) and the definition of $\zeta^{[\geq 1]}$ in (5.25), it follows that

$$X_n := \sum_{l=1}^n \sum_{m=-l}^l (\mathcal{P}_1^{-1}(\partial_r \zeta^{[\geq 1]}, 0))_E^{(lm)} E^{(lm)}$$

is a sequence of smooth vectorfields compactly supported in $\mathbb{R}^3 \setminus \overline{B_1}$ that converges in $\mathcal{H}_{-5/2}^{w-2}$ to $\mathcal{P}_1^{-1}(\partial_r \zeta^{[\geq 1]}, 0)$ as $n \rightarrow \infty$. By the definition of $\overline{\mathcal{H}}_{-5/2}^{w-2}$, this shows that $\mathcal{P}_1^{-1}(\partial_r \zeta^{[\geq 1]}, 0) \in \overline{\mathcal{H}}_{-5/2}^{w-2}$ and hence finishes the precise estimate of $\zeta^{[\geq 1]}$.

Regularity and boundary control of $u^{[\geq 1]}$. By Proposition C.6, for all $w \geq 2$, the scalar function $u^{[\geq 1]}$ defined in (5.24) is bounded by

$$\|u^{[\geq 1]}\|_{H_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \left\| \frac{1}{2}h + \zeta^{[\geq 1]} \right\|_{\overline{H}_{-5/2}^{w-2}}. \quad (5.39)$$

We show now the improved boundary behaviour

$$u^{[\geq 1]} \in \overline{H}_{-1/2}^w.$$

By Proposition C.9 it suffices to prove the next claim.

Claim 5.12. *Let $w \geq 2$ be an integer. It holds that*

$$\partial_r u^{[\geq 1]}|_{r=1} = 0.$$

First, by (5.39), it holds that for $l \geq 1$, $m \in \{-l, \dots, l\}$,

$$\int_1^\infty \frac{1}{(1+r)^2} (u^{(lm)})^2 dr, \int_1^\infty (\partial_r u^{(lm)})^2 dr, \int_1^\infty (1+r)^2 (\partial_r^2 u^{(lm)})^2 dr < \infty.$$

By Lemma 2.14, it follows that

$$\sup_{r \in [1, \infty)} (1+r)^{-1/2} |u^{(lm)}| < \infty, \quad \sup_{r \in [1, \infty)} (1+r)^{1/2} |\partial_r u^{(lm)}| < \infty. \quad (5.40)$$

We show now that for $w \geq 2$ and $l \geq 1$, $m \in \{-l, \dots, l\}$,

$$\partial_r u^{(lm)}|_{r=1} = 0.$$

Definition (5.24) is in the Hodge-Fourier formalism equivalent to the following ODEs for $u^{(lm)}$ with $l \geq 1, m \in \{-l, \dots, l\}$, see Lemma 2.33,

$$\begin{cases} r^{-1+\sqrt{l(l+1)/2}} \partial_r \left(r^{1-2\sqrt{l(l+1)/2}} \partial_r \left(r^{\sqrt{l(l+1)/2}} u^{(lm)} \right) \right) = \frac{1}{2} \left(\frac{1}{2} h^{(lm)} + \zeta^{(lm)} \right), \\ u^{(lm)}|_{r=1} = 0. \end{cases} \quad (5.41)$$

On the one hand,

$$\begin{aligned}
& \int_1^\infty \partial_r \left(r^{1-2\sqrt{l(l+1)/2}} \partial_r \left(r^{\sqrt{l(l+1)/2}} u^{(lm)} \right) \right) dr \\
&= \left[r^{1-\sqrt{l(l+1)/2}} \partial_r u^{(lm)} + (\sqrt{l(l+1)/2}) r^{-\sqrt{l(l+1)/2}} u^{(lm)} \right]_1^\infty \\
&= -\partial_r u^{(lm)}|_{r=1},
\end{aligned}$$

where we used that $l \geq 1$, $u^{(lm)}|_{r=1} = 0$ and (5.40).

On the other hand, by (5.41) and integral identity (5.29),

$$\begin{aligned}
& \int_1^\infty \partial_r \left(r^{1-2\sqrt{l(l+1)/2}} \partial_r \left(r^{\sqrt{l(l+1)/2}} u^{(lm)} \right) \right) dr \\
&= \frac{1}{2} \int_1^\infty r^{1-\sqrt{l(l+1)/2}} \left(\frac{1}{2} h^{(lm)} + \zeta^{(lm)} \right) dr \\
&= 0.
\end{aligned}$$

This shows that for $l \geq 1$, $m \in \{-l, \dots, l\}$,

$$\partial_r u^{(lm)}|_{r=1} = 0.$$

This proves Claim 5.12 and finishes the control of $u^{[\geq 1]}$. This finishes the proof of Proposition 5.10. \square

The next proposition shows that $u^{[0]} \in \overline{H}_{-1/2}^w$ and $\xi \in \overline{\mathcal{H}}_{-1/2}^w$ with quantitative estimates.

Proposition 5.13. *Let $w \geq 2$ be an integer and $h \in \overline{H}_{-5/2}^{w-2}$. Then, the following holds.*

- **Regularity and boundary behaviour of $u^{[0]}$.** *The radial scalar function $u^{[0]}$ defined in (5.23) is bounded by*

$$\|u^{[0]}\|_{H_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|h^{[0]}\|_{\overline{H}_{-5/2}^{w-2}}. \quad (5.42)$$

Furthermore, it holds that $u^{[0]} \in \overline{H}_{-1/2}^w$.

- **Regularity and boundary behaviour of ξ .** *The S_r -tangent vector field ξ defined in (5.27) is bounded by*

$$\|\xi\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|h^{[\geq 1]}\|_{\overline{H}_{-5/2}^{w-2}}. \quad (5.43)$$

Furthermore, it holds that $\xi \in \overline{\mathcal{H}}_{-1/2}^w$.

Proof of Proposition 5.13. We prove each part separately.

Regularity and boundary behaviour of $u^{[0]}$. We first show that for $w \geq 2$,

$$\|u^{[0]}\|_{H_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|h^{[0]}\|_{\overline{H}_{-5/2}^{w-2}}. \quad (5.44)$$

Recall that $u^{[0]}$ satisfies on $r > 1$ by (5.23)

$$\begin{cases} \frac{1}{r^2} \partial_r (r \partial_r (r u^{[0]})) = \frac{1}{4} h^{[0]}, \\ u^{[0]}|_{r=1} = \partial_r u^{[0]}|_{r=1} = 0. \end{cases}$$

By integration, it follows that

$$u^{[0]}(r) = \frac{1}{r} \int_1^r \frac{1}{r'} \left(\int_1^{r'} (r''^2) h^{[0]} dr'' \right) dr'. \quad (5.45)$$

We now prove the case $w = 2$ of (5.44) by showing

$$\|u^{[0]}\|_{H_{-1/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|h^{[0]}\|_{\overline{H}_{-5/2}^0}, \quad (5.46)$$

$$\|\partial_r u^{[0]}\|_{H_{-3/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|h^{[0]}\|_{\overline{H}_{-5/2}^0}, \quad (5.47)$$

$$\|\partial_r^2 u^{[0]}\|_{H_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|h^{[0]}\|_{\overline{H}_{-5/2}^0}. \quad (5.48)$$

Indeed, see Lemma 2.35 and note that tangential regularity follows because $u^{[0]}$ is constant on spheres.

To prove (5.46), use (5.45) and that $u^{[0]}$ is constant on spheres,

$$\begin{aligned} \|u^{[0]}\|_{H_{-1/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2 &= \int_1^\infty \frac{1}{r^2} \left(\int_1^r \frac{1}{r'} \left(\int_1^{r'} (r''^2) h^{[0]} dr'' \right) dr' \right)^2 dr \\ &= \left[-\frac{1}{r} \left(\int_1^r \frac{1}{r'} \left(\int_1^{r'} (r''^2) h^{[0]} dr'' \right) dr' \right)^2 \right]_1^\infty \\ &\quad + 2 \int_1^\infty \frac{1}{r} \left(\frac{1}{r} \int_1^r (r''^2) h^{[0]} dr'' \right) \left(\int_1^r \frac{1}{r'} \left(\int_1^{r'} (r''^2) h^{[0]} dr'' \right) dr' \right) dr. \end{aligned}$$

The boundary term has negative sign and may thus be discarded. The integral term can be estimated by Cauchy-Schwarz as

$$\begin{aligned} & \int_1^\infty \left(\frac{1}{r} \int_1^r (r'^2) h^{[0]} dr' \right) \left(\frac{1}{r} \int_1^r \frac{1}{r'} \left(\int_1^{r'} (r''^2) h^{[0]} dr'' \right) dr' \right) dr \\ & \leq \left(\int_1^\infty \frac{1}{r^2} \left(\int_1^r (r')^2 h^{[0]} dr' \right)^2 dr \right)^{1/2} \|u^{[0]}\|_{H_{-1/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}. \end{aligned}$$

This proves that

$$\|u^{[0]}\|_{H_{-1/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2 \lesssim \int_1^\infty \frac{1}{r^2} \left(\int_1^r (r')^2 h^{[0]} dr' \right)^2 dr. \quad (5.49)$$

A similar integration by parts shows that

$$\begin{aligned} \int_1^\infty \frac{1}{r^2} \left(\int_1^r (r')^2 h^{[0]} dr' \right)^2 dr & \lesssim \int_1^\infty r^4 (h^{[0]})^2 dr \\ & = \|h^{[0]}\|_{\overline{H}_{-5/2}^0}^2. \end{aligned} \quad (5.50)$$

Together, (5.49) and (5.50) prove (5.46).

We now prove (5.47). By differentiating (5.45) in r , it follows that on $r > 1$

$$\partial_r u^{[0]} = -\frac{1}{r} u^{[0]} + \frac{1}{r^2} \int_1^r (r')^2 h^{[0]} dr'.$$

Therefore, by using (5.46) and (5.50),

$$\begin{aligned} \|\partial_r u^{[0]}\|_{H_{-3/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} & \leq \|u^{[0]}\|_{H_{-1/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} + \left\| \frac{1}{r^2} \int_1^r (r')^2 h^{[0]} dr' \right\|_{H_{-3/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \\ & = \|u^{[0]}\|_{H_{-1/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} + \left\| \frac{1}{r} \int_1^r (r')^2 h^{[0]} dr' \right\|_{H_{-1/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \\ & \lesssim \|h^{[0]}\|_{H_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}. \end{aligned}$$

This proves (5.47).

By the defining ODE (5.23), the previous estimates (5.46), (5.47) imply (5.48). This finishes the proof of (5.44) in the case $w = 2$.

We turn now to the case $w > 2$ of (5.44). Higher radial derivatives can be estimated by differentiating the defining ODE (5.23) in r . Tangential regularity is trivial because $u^{[0]}$ is radial. This proves (5.44) for $w \geq 2$.

It remains to show that $u^{[0]} \in \overline{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})$. Indeed, this follows by (5.23) and Proposition 2.13. This finishes the control of $u^{[0]}$.

Regularity and boundary behaviour of ξ . We prove now that for $w \geq 2$

$$\|\xi\|_{\mathcal{H}_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|h^{[\geq 1]}\|_{\overline{H}_{-5/2}^{w-2}}. \quad (5.51)$$

First, we claim that

$$\|\xi\|_{\mathcal{H}_{-1/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2 + \|\nabla \xi\|_{\mathcal{H}_{-3/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2 \lesssim \|h^{[\geq 1]}\|_{\overline{H}_{-5/2}^0}^2. \quad (5.52)$$

Indeed, by (5.27), ξ solves on each S_r , $r \geq 1$,

$$\begin{aligned} \mathrm{d}\!\!\!\!\!/\xi &= \frac{1}{r^3} \int_1^r (r')^3 \zeta^{[\geq 1]} dr', \\ \mathrm{curl} \xi &= 0. \end{aligned}$$

Therefore, by Proposition 2.23, for all $r \geq 1$,

$$\int_{S_r} |\nabla \xi|^2 + \frac{1}{r^2} |\xi|^2 = \int_{S_r} (\mathrm{d}\!\!\!\!\!/\xi)^2. \quad (5.53)$$

We can estimate

$$\begin{aligned}
& \|\xi\|_{\mathcal{H}_{-1/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2 + \|\nabla \xi\|_{\mathcal{H}_{-3/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2 \\
&= \int_1^\infty \int_{S_r} \frac{1}{r^6} \left(\int_1^r (r')^3 \zeta^{[\geq 1]} \right)^2 dr \\
&= \int_{\mathbb{S}^2} \int_1^\infty \frac{1}{r^4} \left(\int_1^r (r')^3 \zeta^{[\geq 1]} dr' \right)^2 dr \\
&= \int_{\mathbb{S}^2} \left[-\frac{1}{3r^3} \left(\int_1^r (r')^3 \zeta^{[\geq 1]} dr' \right)^2 \right]_1^\infty \\
&\quad + \frac{2}{3} \int_{\mathbb{S}^2} \int_1^\infty \frac{1}{r^3} (r^3 \zeta^{[\geq 1]}) \left(\int_1^r (r')^3 \zeta^{[\geq 1]} dr' \right) dr.
\end{aligned}$$

The first term on the right-hand side is non-positive and discarded. The second term can be estimated by Cauchy-Schwarz as

$$\begin{aligned}
& \int_{\mathbb{R}^3 \setminus \overline{B_1}} \frac{1}{r^5} (r^3 \zeta^{[\geq 1]}) \left(\int_1^r (r')^3 \zeta^{[\geq 1]} dr' \right) \\
&\leq \left(\int_{\mathbb{R}^3 \setminus \overline{B_1}} \frac{1}{r^6} \left(\int_1^r (r')^3 \zeta^{[\geq 1]} \right)^2 \right)^{1/2} \left(\int_{\mathbb{R}^3 \setminus \overline{B_1}} r^2 (\zeta^{[\geq 1]})^2 \right)^{1/2} \\
&= \left(\int_{\mathbb{R}^3 \setminus \overline{B_1}} (\operatorname{div} \xi)^2 \right)^{1/2} \left(\int_{\mathbb{R}^3 \setminus \overline{B_1}} r^2 (\zeta^{[\geq 1]})^2 \right)^{1/2} \\
&= \left(\|\xi\|_{\mathcal{H}_{-1/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2 + \|\nabla \xi\|_{\mathcal{H}_{-3/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2 \right)^{1/2} \|\zeta^{[\geq 1]}\|_{\overline{H}_{-5/2}^0},
\end{aligned}$$

where we used (5.53). This shows that

$$\|\xi\|_{\mathcal{H}_{-1/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} + \|\nabla \xi\|_{\mathcal{H}_{-3/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\zeta^{[\geq 1]}\|_{\overline{H}_{-5/2}^0}.$$

By Proposition 5.10, this proves (5.52).

Next, we consider radial regularity of order 1. We claim that

$$\|\nabla_N \xi\|_{\mathcal{H}_{-3/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2 + \|\nabla \nabla_N \xi\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2 \lesssim \|h^{[\geq 1]}\|_{\overline{H}_{-5/2}^0}^2. \quad (5.54)$$

Indeed, by Lemma 2.35, it follows that on $r > 1$

$$\begin{aligned} r \operatorname{div} \nabla_N \xi &= \partial_r (r \operatorname{div} \xi) \\ &= \partial_r \left(\frac{1}{r^2} r^3 \operatorname{div} \xi \right) \\ &= -2 \operatorname{div} \xi + r \left(\frac{1}{r^3} \partial_r (r^3 \operatorname{div} \xi) \right), \\ \operatorname{curl} \nabla_N \xi &= 0 \end{aligned}$$

Therefore $\nabla_N \xi$ solves on each S_r , $r \geq 1$, the Hodge system

$$\begin{aligned} \operatorname{div} \nabla_N \xi &= -\frac{2}{r} \operatorname{div} \xi + \zeta^{[\geq 1]}, \\ \operatorname{curl} \nabla_N \xi &= 0. \end{aligned}$$

By Propositions 2.23 and 5.10, this proves (5.54).

Similarly, we have the higher radial regularity for each $w \geq 2$,

$$\|\nabla_N^w \xi\|_{\mathcal{H}_{-1/2-w}^0(\mathbb{R}^3 \setminus \overline{B_1})}^2 \lesssim \|h^{[\geq 1]}\|_{\overline{H}_{-5/2}^{w-2}}^2. \quad (5.55)$$

This follows by an induction in $w \geq 2$, using that by

$$\begin{aligned} \operatorname{div} (\nabla_n^w \xi) &= \frac{1}{r} \partial_r^w (r \operatorname{div} \xi) \\ &= \frac{1}{r} \partial_r^{w-1} (-2 \operatorname{div} \xi + r \zeta^{[\geq 2]}) \\ &= -2 \operatorname{div} \left(\nabla_N^{w-1} \left(\frac{1}{r} \xi \right) \right) + \frac{1}{r} \partial_r^{w-1} \left(\frac{1}{r} \zeta^{[\geq 1]} \right) \end{aligned}$$

so that we have

$$\nabla_n^w \xi = -2 \nabla_n^{w-1} \left(\frac{1}{r} \xi \right) + \nabla_N^{w-2} \left(\frac{1}{r} \mathcal{P}_1^{-1}(\zeta^{[\geq 1]}, 0) \right) \quad (5.56)$$

$$+ \nabla_N^{w-2} (\mathcal{P}_1^{-1}(\partial_r \zeta^{[\geq 1]}, 0)), \quad (5.57)$$

where the last term is controlled by Proposition 5.10.

We turn now to tangential regularity. We claim first that

$$\|\nabla \nabla \xi\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|h^{[\geq 1]}\|_{\overline{H}_{-5/2}^0}, \quad (5.58)$$

By Proposition 2.34, it suffices to estimate for $l \geq 1, m \in \{-l, \dots, l\}$

$$\int_1^\infty r^2 \left(\frac{l(l+1)}{r^2} \xi_E^{(lm)} \right)^2 dr \lesssim \int_1^\infty r^2 (h^{(lm)})^2 dr, . \quad (5.59)$$

Definition (5.27) is in the Hodge-Fourier formalism equivalent to the following expression for $\xi_E^{(lm)}$, $l \geq 1, m \in \{-l, \dots, l\}$,

$$\frac{\sqrt{l(l+1)}}{r} \xi_E^{(lm)} = \frac{1}{r^2} \left(\int_1^r (r')^2 \zeta^{(lm)} dr' \right). \quad (5.60)$$

Rewrite first

$$\begin{aligned} \int_1^\infty r^2 \left(\frac{l(l+1)}{r^2} \right)^2 \left(\xi_E^{(lm)} \right)^2 dr &= l(l+1) \int_1^\infty \frac{1}{r^4} \left(\int_1^r (r')^2 \zeta^{(lm)} dr' \right)^2 dr \\ &= l(l+1) \int_1^{1+\frac{1}{l}} \frac{1}{r^4} \left(\int_1^r (r')^2 \zeta^{(lm)} dr' \right)^2 dr, \end{aligned} \quad (5.61)$$

where in the last integral we bounded the domain of integration by combining the integral identity (5.30) and the fact that

$$\text{supp} \zeta^{(lm)} \subset \left[1, 1 + \frac{1}{l} \right].$$

By (5.25) and (5.26), the right-hand side of (5.61) can be estimated by

$$\begin{aligned} &l(l+1) \int_1^{1+\frac{1}{l}} \frac{1}{r^4} \left(\int_1^r (r')^2 \zeta^{(lm)} dr' \right)^2 dr \\ &\lesssim l(l+1) \int_1^{1+\frac{1}{l}} \frac{1}{r^4} \left(\int_1^r c^{(lm)}(r') \sqrt{l(l+1)/2+1} \partial_r (\chi(l(r-1))) dr' \right)^2 dr \\ &\quad + l(l+1) \int_1^{1+\frac{1}{l}} \frac{1}{r^4} \left(\int_1^r \tilde{c}^{(lm)}(r') \sqrt{l(l+1)/2+1} \partial_r^2 (\chi(l(r-1))) dr' \right)^2 dr. \end{aligned} \quad (5.62)$$

The first term on the right-hand side is estimated as

$$\begin{aligned}
& l(l+1) \int_1^{1+\frac{1}{l}} \frac{1}{r^4} \left(\int_1^r c^{(lm)}(r') \sqrt{l(l+1)/2+1} \partial_r (\chi(l(r-1))) dr' \right)^2 dr \\
& \lesssim l(l+1) \int_1^{1+\frac{1}{l}} \frac{1}{r^4} (c^{(lm)})^2 l^2 r^{2\sqrt{l(l+1)/2+2}} \left(\int_1^{1+\frac{1}{l}} dr' \right)^2 dr \\
& \lesssim l(l+1) (c^{(lm)})^2 \int_1^{1+\frac{1}{l}} r^{2\sqrt{l(l+1)/2-2}} dr \\
& \lesssim \int_1^\infty (rh^{(lm)})^2 dr,
\end{aligned}$$

where we used (5.33) and the fact that $l \geq 1$. The second term on the right-hand side of (5.62) is estimated similarly by using (5.34), this is left to the reader. This proves (5.59) and therefore (5.58).

We also have the higher tangential regularity

$$\|\nabla^w \xi\|_{\mathcal{H}_{-1/2-w}^0(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|h^{[\geq 1]}\|_{\overline{H}_{-5/2}^{w-2}} \quad (5.63)$$

Indeed, in the Hodge-Fourier formalism, see (5.60),

$$\begin{aligned}
\left| \left(\frac{\sqrt{l(l+1)}}{r} \right)^w \xi_E^{(lm)} \right| &= \frac{1}{\sqrt{l(l+1)}r} \left| \int_1^r (r')^2 \left(\frac{\sqrt{l(l+1)}}{r} \right)^w \zeta^{(lm)} dr' \right| \\
&\lesssim \frac{1}{\sqrt{l(l+1)}r} \left| \int_1^r (r')^2 \left(\frac{\sqrt{l(l+1)}}{r'} \right)^w \zeta^{(lm)} dr' \right|.
\end{aligned}$$

Together with the higher regularity of $\zeta^{[\geq 1]}$ provided by Proposition 5.10, similar estimates as for (5.58) imply (5.63).

To summarise, (5.52), (5.54), (5.55), (5.58) and (5.63) imply (5.51) for $w \geq 2$.

It remains to show that $\xi \in \overline{\mathcal{H}}_{-1/2}^w$. This follows by induction from (5.56) and the fact that $\zeta^{[\geq 1]}, \mathcal{P}_1^{-1}(\partial_r \zeta^{[\geq 1]}, 0) \in \overline{\mathcal{H}}_{-5/2}^{w-2}$ together with Proposition 2.13. This finishes the proof of Proposition 5.13. \square

6. PROOF OF THE MAIN THEOREM 3.1

In this section, we prove the Main Theorem 3.1. The idea of the proof is to use Theorems 4.1 and 5.1 to set up an iterative scheme. We show that this scheme is well-defined and converges to a fixpoint which solves the maximal constraint equations on \mathbb{R}^3 .

6.1. Setup of the iterative scheme. In this section, we define a sequence of pairs $(g_i, k_i)_{i \geq 1}$, where for each $i \geq 1$, g_i is an $\mathcal{H}_{-1/2}^w$ -asymptotically flat metric and $k_i \in \mathcal{H}_{-3/2}^{w-1}$ a symmetric 2-tensor on \mathbb{R}^3 .

Let $\varepsilon > 0$ be a small constant to be determined later. Let $(\bar{g}, \bar{k}) \in \mathcal{H}^w(B_1) \times \mathcal{H}^{w-1}(B_1)$ be a solution to the maximal constraint equations on B_1 ,

$$\begin{aligned} R(\bar{g}) &= |\bar{k}|_{\bar{g}}^2, \\ \operatorname{div}_{\bar{g}} \bar{k} &= 0, \\ \operatorname{tr}_{\bar{g}} \bar{k} &= 0 \end{aligned} \tag{6.1}$$

such that

$$\|(\bar{g} - e, \bar{k})\|_{\mathcal{H}^w(B_1) \times \mathcal{H}^{w-1}(B_1)} < \varepsilon.$$

- **Definition of (g_1, k_1) .** Use standard Sobolev extension to extend \bar{g} from B_1 to an \mathcal{H}_δ^w -asymptotically flat metric g_1 on \mathbb{R}^3 such that

$$\|g_1 - e\|_{\mathcal{H}_{-1/2}^w} \lesssim \|\bar{g} - e\|_{\mathcal{H}^w(B_1)}.$$

Similarly, extend \bar{k} from B_1 to a symmetric g_1 -tracefree 2-tensor $k_1 \in \mathcal{H}_{-3/2}^{w-1}$ such that

$$\|k_1\|_{\mathcal{H}_{-3/2}^{w-1}} \lesssim \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)}, \tag{6.2}$$

see Lemma 4.4.

- **Definition of $(g_{i+1} - e, k_{i+1})$ for $i \geq 1$.** Given (g_i, k_i) , define (g_{i+1}, k_{i+1}) as follows.

First, let g_{i+1} be the $\mathcal{H}_{-1/2}^w$ -asymptotically flat metric on \mathbb{R}^3 constructed by Theorem 5.1 such that

$$\begin{aligned} g_{i+1}|_{B_1} &= \bar{g}, \\ R(g_{i+1}) &= |k_i|_{g_i}^2 \text{ on } \mathbb{R}^3. \end{aligned}$$

Here we assumed that $\|(g_i - e, k_i)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}}$ is sufficiently small.

Second, let $k_{i+1} \in \mathcal{H}_{-3/2}^{w-1}$ be the symmetric 2-tensor on \mathbb{R}^3 constructed by Theorem 4.1 such that

$$k_{i+1}|_{B_1} = \bar{k}$$

and on \mathbb{R}^3

$$\begin{aligned}\operatorname{div}_{g_{i+1}}(k_{i+1}) &= 0, \\ \operatorname{tr}_{g_{i+1}}(k_{i+1}) &= 0.\end{aligned}$$

Here we assumed further that $\|g_{i+1} - e\|_{\mathcal{H}_{-1/2}^w}$ is sufficiently small.

In the next section we prove that for $\varepsilon > 0$ small enough, the above smallness assumptions are satisfied for all $i \geq 1$. In particular, that the sequence is well-defined.

6.2. Convergence of the iterative scheme. In this section, we combine the iteration estimates of Theorems 4.1 and 5.1 to prove estimates for the iteration scheme defined above in Section 6.1. These estimates then directly imply uniform boundedness and convergence of the sequence $(g_i, k_i)_{i \geq 1}$.

The next proposition is the main result of this section.

Proposition 6.1. *Let $w \geq 2$ be an integer. There exist universal constants $\varepsilon > 0, c > 0$ such that if for an $i \geq 2$ it holds that*

$$\|(g_i - e, k_i)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} < \varepsilon, \|(g_{i-1} - e, k_{i-1})\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} < \varepsilon,$$

then

$$\begin{aligned}& \|(g_{i+1} - g_i, k_{i+1} - k_i)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} \\ & \leq c \left(\|(g_i - e, k_i)\|_{\mathcal{H}_{-1/2}^w} + \|(g_{i-1} - e, k_{i-1})\|_{\mathcal{H}_{-1/2}^w} + \|(g_{i-1} - e, k_{i-1})\|_{\mathcal{H}_{-1/2}^w}^2 \right) \\ & \quad \times \|(g_i - g_{i-1}, k_i - k_{i-1})\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}}.\end{aligned} \tag{6.3}$$

Before proving Proposition 6.1, we state the following technical lemma. Its proof is based on Lemma 2.9 and Corollary 2.11 and left to the reader, see also Lemma 2.38.

Lemma 6.2. *Let $w \geq 2$ be an integer. Let g and g' be two $\mathcal{H}_{-1/2}^w$ -asymptotically flat metrics on \mathbb{R}^3 . There exists universal constant $\varepsilon > 0$ such that if*

$$\|g - e\|_{\mathcal{H}_{-1/2}^w} < \varepsilon,$$

then for all symmetric 2-tensors $V \in \mathcal{H}_{-3/2}^{w-1}$,

$$\| |V|_g^2 - |V|_{g'}^2 \|_{H_{-5/2}^{w-2}} \lesssim \|g - g'\|_{\mathcal{H}_{-1/2}^w} \|V\|_{\mathcal{H}_{-3/2}^{w-1}}^2.$$

We turn now to the proof of Proposition 6.1.

Proof of Proposition 6.1. On the one hand, for $\varepsilon > 0$ sufficiently small, it follows by the iteration estimates of Theorem 5.1 and Lemmas 6.2 that

$$\begin{aligned}
\|g_{i+1} - g_i\|_{\mathcal{H}_{-1/2}^w} &\lesssim \|R(g_{i+1}) - R(g_i)\|_{H_{-5/2}^{w-2}} \\
&\lesssim \| |k_i|_{g_i}^2 - |k_{i-1}|_{g_{i-1}}^2 \|_{H_{-5/2}^{w-2}(\mathbb{R}^3)} \\
&\lesssim \| |k_i|_{g_i}^2 - |k_{i-1}|_{g_i}^2 \|_{H_{-5/2}^{w-2}(\mathbb{R}^3)} + \| |k_{i-1}|_{g_i}^2 - |k_{i-1}|_{g_{i-1}}^2 \|_{H_{-5/2}^{w-2}(\mathbb{R}^3)} \\
&\lesssim \| |k_i|_{g_i}^2 - |k_{i-1}|_{g_i}^2 \|_{H_{-5/2}^{w-2}(\mathbb{R}^3)} + \|k_{i-1}\|_{\mathcal{H}_{-3/2}^{w-1}}^2 \|g_i - g_{i-1}\|_{\mathcal{H}_{-1/2}^w}
\end{aligned} \tag{6.4}$$

Using Lemma 2.8 and the identity

$$|k_i|_{g_i}^2 - |k_{i-1}|_{g_i}^2 = (k_i - k_{i-1})^{ab} (k_i + k_{i-1})_{ab},$$

we have, for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned}
\| |k_i|_{g_i}^2 - |k_{i-1}|_{g_i}^2 \|_{H_{-5/2}^{w-2}(\mathbb{R}^3)} &\lesssim \|k_i + k_{i-1}\|_{\mathcal{H}_{-3/2}^{w-1}} \|k_i - k_{i-1}\|_{\mathcal{H}_{-3/2}^{w-1}} \\
&\lesssim \left(\|k_i\|_{\mathcal{H}_{-3/2}^{w-1}} + \|k_{i-1}\|_{\mathcal{H}_{-3/2}^{w-1}} \right) \|k_i - k_{i-1}\|_{\mathcal{H}_{-3/2}^{w-1}}.
\end{aligned}$$

Plugging this into (6.4), we get

$$\begin{aligned}
&\|g_{i+1} - g_i\|_{\mathcal{H}_{-1/2}^w} \\
&\lesssim \left(\|k_i\|_{\mathcal{H}_{-3/2}^{w-1}} + \|k_{i-1}\|_{\mathcal{H}_{-3/2}^{w-1}} + \|k_{i-1}\|_{\mathcal{H}_{-3/2}^{w-1}}^2 \right) \|(g_i, k_i) - (g_{i-1}, k_{i-1})\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}}.
\end{aligned}$$

On the other hand, for $\varepsilon > 0$ sufficiently small, by the iteration estimates of Theorem 4.1,

$$\begin{aligned}
\|k_{i+1} - k_i\|_{\mathcal{H}_{-3/2}^{w-1}} &\lesssim \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)} \|g_{i+1} - g_i\|_{\mathcal{H}_{-1/2}^w} \\
&\lesssim \|k_i\|_{\mathcal{H}_{-3/2}^{w-1}} \|g_{i+1} - g_i\|_{\mathcal{H}_{-1/2}^w}.
\end{aligned}$$

Combining the two last estimates, it follows that for $\varepsilon > 0$ sufficiently small, there exists a universal constant $c > 0$ such that

$$\begin{aligned}
&\|(g_{i+1}, k_{i+1}) - (g_i, k_i)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} \\
&\leq c \left(\|k_i\|_{\mathcal{H}_{-3/2}^{w-1}} + \|k_{i-1}\|_{\mathcal{H}_{-3/2}^{w-1}} + \|k_{i-1}\|_{\mathcal{H}_{-3/2}^{w-1}}^2 \right) \|(g_i, k_i) - (g_{i-1}, k_{i-1})\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}}.
\end{aligned}$$

This finishes the proof of Proposition 6.1. \square

We are now in position to prove that the sequence is well-defined and derive a uniform estimate.

Lemma 6.3. *Let $w \geq 2$ be an integer. There is a universal $\varepsilon > 0$ small enough such that if*

$$\|(\bar{g} - e, \bar{k})\|_{\mathcal{H}^w(B_1) \times \mathcal{H}^{w-1}(B_1)} < \varepsilon,$$

then the sequence $(g_i, k_i)_{i \geq 1}$ is well-defined and for all $i \geq 2$,

$$\|(g_{i+1}, k_{i+1}) - (g_i, k_i)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} \leq \frac{1}{4} \|(g_i - g_{i-1}, k_i - k_{i-1})\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}}. \quad (6.5)$$

Furthermore, it is uniformly bounded by

$$\|(g_i, k_i)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} \lesssim \|(\bar{g} - e, \bar{k})\|_{\mathcal{H}^w(B_1) \times \mathcal{H}^{w-1}(B_1)}.$$

Proof. The proof of (6.5) goes by induction in $i \geq 2$.

The case $i = 2$. By construction, for $\varepsilon > 0$ sufficiently small, by Theorems 4.1 and 5.1,

$$\begin{aligned} \|(g_1 - e, k_1)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} &\lesssim \|(\bar{g} - e, \bar{k})\|_{\mathcal{H}^w(B_1) \times \mathcal{H}^{w-1}(B_1)}, \\ \|(g_2 - e, k_2)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} &\lesssim \|\bar{g} - e\|_{\mathcal{H}^w(B_1)} + \|R(g_2)\|_{H_{-5/2}^{w-2}} + \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)} \\ &\lesssim \|\bar{g} - e\|_{\mathcal{H}^w(B_1)} + \|k_1\|_{g_1}^2 \|H_{-5/2}^{w-2}\| + \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)} \\ &\lesssim \|\bar{g} - e\|_{\mathcal{H}^w(B_1)} + \|k_1\|_{\mathcal{H}_{-3/2}^{w-1}}^2 + \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)} \\ &\lesssim \|(\bar{g} - e, \bar{k})\|_{\mathcal{H}^w(B_1) \times \mathcal{H}^{w-1}(B_1)}, \end{aligned}$$

where we used Lemmas 2.8 and 2.9. By Theorems 4.1 and 5.1, for $\varepsilon > 0$ sufficiently small, (g_3, k_3) is well-defined.

By Proposition 6.1, there exists a universal $c > 0$ such that

$$\begin{aligned} &\|(g_3 - g_2, k_3 - k_2)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} \\ &\leq 3c \|(\bar{g} - e, \bar{k})\|_{\mathcal{H}^w(B_1) \times \mathcal{H}^{w-1}(B_1)} \|(g_2 - g_1, k_2 - k_1)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} \end{aligned}$$

Let $\varepsilon < \frac{1}{24c}$. This proves the case $i = 2$.

The induction step $i \rightarrow i + 1$. Using the induction hypothesis, it holds for $j = i - 1, i$ that

$$\begin{aligned} &\|(g_{j+1} - e, k_{j+1})\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} \\ &\leq \|(g_{j+1} - g_j, k_{j+1} - k_j)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} + \cdots + \|(g_3 - g_2, k_3 - k_2)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} \\ &\quad + \|(g_2 - e, k_2)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} \\ &\leq \sum_{m=1}^{j-2} \frac{1}{4^m} \|(g_2 - g_1, k_2 - k_1)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} + \|(g_2 - e, k_2)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} \\ &\leq 2 \|(g_2 - g_1, k_2 - k_1)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} + \|(g_2 - e, k_2)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} \\ &\lesssim \|(\bar{g} - e, \bar{k})\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}}. \end{aligned} \quad (6.6)$$

This shows that (g_{i+2}, k_{i+2}) is well-defined for $\varepsilon > 0$ sufficiently small. By applying Proposition 6.1, there exists a universal constant $c' > 0$ such that

$$\|(g_{i+2} - g_{i+1}, k_{i+2} - k_{i+1})\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} \leq c' \varepsilon \|(g_{i+1} - g_i, k_{i+1} - k_i)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}}.$$

Let $\varepsilon < \frac{1}{24c'}$. This finishes the induction step and hence the proof of (6.5).

For $\varepsilon > 0$ small, we have, as in (6.6), the uniform estimate for all $i \geq 1$,

$$\|(g_i, k_i)\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} \lesssim \|(\bar{g} - e, \bar{k})\|_{\mathcal{H}^w(B_1) \times \mathcal{H}^{w-1}(B_1)}.$$

This finishes the proof of Lemma 6.3. □

Lemma 6.3 implies convergence of the iterative scheme.

Corollary 6.4. *There exists $\varepsilon > 0$ small such that if*

$$\|(\bar{g}, \bar{k})\|_{\mathcal{H}^w(B_1) \times \mathcal{H}^{w-1}(B_1)} < \varepsilon,$$

then the sequence $(g_i - e, k_i)_{i \geq 1}$ converges in $\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}$ as $i \rightarrow \infty$. Its limit

$$(g', k') := \lim_{i \rightarrow \infty} (g_i, k_i) \in \mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}$$

solves the maximal constraint equations on \mathbb{R}^3

$$\begin{aligned} R(g') &= |k'|_{g'}^2, \\ \operatorname{div}_{g'} k' &= 0, \\ \operatorname{tr}_{g'} k' &= 0 \end{aligned} \tag{6.7}$$

and satisfies $(g', k')|_{B_1} = (\bar{g}, \bar{k})$. Moreover,

$$\|(g' - e, k')\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} \lesssim \|(\bar{g} - e, \bar{k})\|_{\mathcal{H}^w(B_1) \times \mathcal{H}^{w-1}(B_1)}.$$

Proof of Corollary 6.4. Lemma 6.3 shows that the iterative scheme $(g_i, k_i)_{i \geq 1}$ is a contraction in the Hilbert space $\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}$. By the Banach fixpoint theorem, the scheme therefore converges to a fixpoint,

$$(g', k') := \lim_{i \rightarrow \infty} (g_i, k_i) \in \mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}.$$

The convergence in $\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}$ is strong enough to conclude that (g', k') solves (6.7) and moreover, by construction of the sequence,

$$(g', k')|_{B_1} = (\bar{g}, \bar{k}).$$

By the uniform estimate in Lemma 6.3,

$$\|(g' - e, k')\|_{\mathcal{H}_{-1/2}^w \times \mathcal{H}_{-3/2}^{w-1}} \lesssim \|(\bar{g} - e, \bar{k})\|_{\mathcal{H}^w(B_1) \times \mathcal{H}^{w-1}(B_1)}.$$

This finishes the proof of Corollary 6.4 □

APPENDIX A. THE PROOF OF PROPOSITION 2.30

In this section we prove Proposition 2.30. First we show that

$$\left\{ E^{(lm)}, H^{(lm)} : l \geq 1, m \in \{-l, \dots, l\} \right\}$$

is a complete orthonormal basis for L^2 -integrable vectorfields on $(S_r, \overset{\circ}{\gamma})$, $r > 0$. The orthonormality of $E^{(lm)}, H^{(lm)}$ defined in (2.10) follows from the orthonormality and completeness of the spherical harmonics $Y^{(lm)}$. Indeed, by (2.10), for all index pairs $(lm), (l'm')$,

$$\begin{aligned} \int_{S_r} E^{(lm)} \cdot E^{(l'm')} &= \frac{r^2}{l(l+1)} \int_{S_r} (Y^{(lm)}, 0) \cdot (\mathcal{D}_1 \mathcal{D}_1^*) (Y^{(l'm')}, 0) \\ &= \frac{r^2}{l(l+1)} \int_{S_r} (Y^{(lm)}, 0) \cdot (-\Delta Y^{(l'm')}, 0) \\ &= \int_{S_r} Y^{(lm)} Y^{(l'm')} \\ &= \delta^{ll'} \delta^{mm'}, \end{aligned}$$

where we used Lemma 2.25 and denoted the pointwise product of two pairs of functions

$$(f_1, f_2) \cdot (f_3, f_4) := (f_1 f_3, f_2 f_4).$$

The same holds for $H^{(lm)}$. Furthermore, for all index pairs $(lm), (l'm')$,

$$\begin{aligned} \int_{S_r} E^{(lm)} \cdot H^{(l'm')} &= \int_{S_r} (Y^{(lm)}, 0) \cdot (0, Y^{(l'm')}) \\ &= 0. \end{aligned}$$

This proves the orthonormality of the vectorfields $E^{(lm)}, H^{(lm)}$.

We now show that the vectorfields $E^{(lm)}, H^{(lm)}$ form a complete basis of vectorfields in $L^2(S_r)$ for every $r > 0$. It suffices to show that for any vector $Z \in L^2(S_r)$,

$$\left(Z_E^{(lm)} = Z_H^{(lm)} = 0 \text{ for all } l \geq 1, m \in \{-l, \dots, l\} \right) \Rightarrow Z = 0.$$

By the identities of Lemma 2.33, for all $l \geq 1, m \in \{-l, \dots, l\}$,

$$\begin{aligned} 0 = Z_E^{(lm)} &= \int_{S_r} Z \cdot E^{(lm)} = \frac{r}{\sqrt{l(l+1)}} \int_{S_r} (\operatorname{div} Z) Y^{(lm)}, \\ 0 = Z_H^{(lm)} &= \int_{S_r} Z \cdot H^{(lm)} = \frac{r}{\sqrt{l(l+1)}} \int_{S_r} (\operatorname{curl} Z) Y^{(lm)}. \end{aligned}$$

By the completeness of $Y^{(lm)}$, see Lemma 2.27, this shows that

$$\mathrm{div} Z = \mathrm{curl} Z = 0.$$

By the ellipticity of this Hodge system, see Proposition 2.23, it follows that $Z = 0$. This proves the completeness of the basis $E^{(lm)}, H^{(lm)}$, $l \geq 1, m \in \{-l, \dots, l\}$.

Second, we show that

$$\left\{ \psi^{(lm)}, \phi^{(lm)} : l \geq 2, m \in \{-l, \dots, l\} \right\}$$

is a complete orthonormal basis of tracefree symmetric 2-tensors in $L^2(S_r)$, $r > 0$. The orthonormality of the $\psi^{(lm)}, \phi^{(lm)}$ defined in (2.11) is proved analogously to the orthonormality of $E^{(lm)}, H^{(lm)}$ and left to the reader.

To prove the completeness of the $\psi^{(lm)}, \phi^{(lm)}$, we need to prove that for any tracefree symmetric 2-tensor $V \in L^2(S_r)$,

$$\left(V_\psi^{(lm)} = V_\phi^{(lm)} = 0 \text{ for all } l \geq 2, m \in \{-l, \dots, l\} \right) \Rightarrow V = 0.$$

This follows however by the completeness of the $E^{(lm)}, H^{(lm)}$ and Proposition 2.23, similar to the above proof for $E^{(lm)}, H^{(lm)}$. This proves the completeness of the basis $\psi^{(lm)}, \phi^{(lm)}$, $l \geq 2, m \in \{-l, \dots, l\}$.

The equality of norms follows by the orthonormality and completeness properties. This finishes the proof of Proposition 2.30.

APPENDIX B. THE PROOFS OF PROPOSITION 2.34 AND LEMMA 2.35

In this section we prove Proposition 2.34 and Lemma 2.35 .

Proof of Proposition 2.34. Consider the first relation. We show at first that that

$$\|\nabla^n u\|_{L^2(S_r)}^2 \lesssim \sum_{l \geq 0} \sum_{m=-l}^l \left(\frac{l(l+1)}{r^2} \right)^n (u^{(lm)})^2 \quad (\text{B.1})$$

by induction in $n \geq 0$. The cases $n = 0, 1$ are verified by Lemma 2.33 and Propositions 2.23 and 2.30.

For the induction step $n \rightarrow n+1$, integrate by parts to estimate

$$\begin{aligned}
& \|\nabla^{n+1} u\|_{L^2(S_r)}^2 \\
&= - \int_{S_r} \nabla (\nabla^n u) \cdot \nabla^n u \\
&= - \int_{S_r} \nabla^n \nabla u \cdot \nabla^n u + [\nabla, \nabla^n] u \cdot \nabla^n u \\
&\leq \int_{S_r} \nabla^{n-1} \nabla u \cdot \nabla \nabla^{n-1} u + \|[\nabla, \nabla^n] u\|_{L^2(S_r)} \|\nabla^n u\|_{L^2(S_r)} \\
&\lesssim \|\nabla^{n-1} \nabla u\|_{L^2(S_r)} \|\nabla^{n+1} u\|_{L^2(S_r)} + \frac{1}{r^2} \sum_{l \geq 0} \sum_{m=-l}^l \left(\frac{l(l+1)}{r^2} \right)^n (u^{(lm)})^2,
\end{aligned} \tag{B.2}$$

where we used that

$$\begin{aligned}
\|[\nabla, \nabla^n] u\|_{L^2(S_r)}^2 &\lesssim \frac{1}{r^4} \sum_{l \geq 0} \sum_{m=-l}^l \left(\frac{l(l+1)}{r^2} \right)^n (u^{(lm)})^2, \\
\|\nabla^n u\|_{L^2(S_r)}^2 &\lesssim \sum_{l \geq 0} \sum_{m=-l}^l \left(\frac{l(l+1)}{r^2} \right)^n (u^{(lm)})^2
\end{aligned}$$

by the fact that we work on the round sphere $(S_r, \overset{\circ}{\gamma})$ and the induction hypothesis.

It follows by (B.2) via Cauchy with weights that

$$\|\nabla^{n+1} u\|_{L^2(S_r)}^2 \lesssim \|\nabla^{n-1} \nabla u\|_{L^2(S_r)}^2 + \frac{1}{r^2} \sum_{l \geq 0} \sum_{m=-l}^l \left(\frac{l(l+1)}{r^2} \right)^n (u^{(lm)})^2. \tag{B.3}$$

To estimate the first term on the right-hand side, we use the induction assumption and Lemma 2.33,

$$\begin{aligned}
\|\nabla^{n-1} \nabla u\|_{L^2(S_r)}^2 &\lesssim \sum_{l \geq 0} \sum_{m=-l}^l \left(\frac{l(l+1)}{r^2} \right)^{n-1} ((\nabla u)^{(lm)})^2 \\
&= \sum_{l \geq 0} \sum_{m=-l}^l \left(\frac{l(l+1)}{r^2} \right)^{n-1} \left(\frac{l(l+1)}{r^2} u^{(lm)} \right)^2 \\
&= \sum_{l \geq 0} \sum_{m=-l}^l \left(\frac{l(l+1)}{r^2} \right)^{n+1} (u^{(lm)})^2.
\end{aligned}$$

Plugging into (B.3) yields

$$\|\nabla^{n+1} u\|_{L^2(S_r)}^2 \lesssim \sum_{l \geq 0} \sum_{m=-l}^l \left(\frac{l(l+1)}{r^2} \right)^{n+1} (u^{(lm)})^2.$$

This finishes the induction and proves (B.1). The other direction needed for the equivalence relation is proved similarly and left to the reader.

It remains to show the second equivalence relation for a vectorfield X . This follows as for scalar functions by induction on $n \geq 0$, using this time the vectorfields $E^{(lm)}, H^{(lm)}$ with Remark 2.28 and Lemma 2.33. This finishes the proof of Proposition 2.34. \square

Proof of Lemma 2.35. We prove each part separately.

Part (1). The estimate (2.12) follows directly by Definition 2.29 and the fact that $Y^{(lm)} \sim r^{-1}$ due to its normalisation, see Section 2.7.

Part (2). On the one hand, it generally holds that in standard polar frame components, see (2.2), for $A = 1, 2$,

$$\begin{aligned} \partial_r (X^A) &= N (X^A) \\ &= e(\nabla_N X, e_A) - e(X, \nabla_N e_A) \\ &= (\nabla_N X)^A - e(X, \nabla_{e_A} N) - e(X, [N, e_A]) \\ &= (\nabla_N X)^A - X^B (-\Theta_{BA}) + \frac{1}{r} X^A \\ &= (\nabla_N X)^A, \end{aligned} \tag{B.4}$$

where we used that in the Euclidean case, for $A = 1, 2$,

$$[N, e_A] = -\frac{1}{r} e_A$$

and, in standard polar frame components,

$$\Theta_{11} = \Theta_{22} = -\frac{1}{r}, \Theta_{12} = \Theta_{21} = 0.$$

Consequently,

$$\begin{aligned} \partial_r (X_E^{(lm)}) &= \int_{S_r} \partial_r (X^A) (E^{(lm)})_A + \frac{1}{r} X_E^{(lm)} \\ &= (\nabla_N X)_E^{(lm)} + \frac{1}{r} X_E^{(lm)}, \end{aligned}$$

where we used in the first equality that on $(S_r, \overset{\circ}{\gamma})$, $\sqrt{\det \overset{\circ}{\gamma}} \sim r^2$ and that in polar frame components, for $A = 1, 2$, see (2.2),

$$(E^{(lm)})_A = -e_A (Y^{(lm)}) \frac{r}{\sqrt{l(l+1)}} \sim \frac{1}{r}.$$

Repeatedly applying ∇_N then proves the first of (2.13). The second of (2.13) is proved similarly.

On the other hand, it holds generally in standard polar frame components that

$$\begin{aligned} \operatorname{div} X &= e_1 (X^1) + e_2 (X^2) \\ &= \frac{1}{r} \partial_{\theta^1} (X^1) + \frac{1}{r \sin \theta^1} \partial_{\theta^2} (X^2). \end{aligned}$$

This leads to

$$\begin{aligned} \partial_r (r \operatorname{div} X) &= \partial_{\theta^1} \partial_r (X^1) + \frac{1}{\sin \theta^1} \partial_{\theta^2} \partial_r (X^2) \\ &= \partial_{\theta^1} (\nabla_N X)^1 + \frac{1}{\sin \theta^1} \partial_{\theta^2} (\nabla_N X)^2 \\ &= r \operatorname{div} \nabla_N X, \end{aligned}$$

where we used (B.4). This finishes the proof of part (2) of Lemma 2.35.

Part (3). The proof of part (3) is similar to part (2) and left to the interested reader. This finishes the proof of Lemma 2.35. \square

APPENDIX C. ELLIPTIC OPERATORS ON WEIGHTED SOBOLEV SPACES

In this section, we first introduce the weak formulation of boundary value problems in weighted spaces. Second, we prove ellipticity and derive higher elliptic regularity estimates in $H_\delta^w(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{H}_\delta^1$ and \overline{H}_δ^w for PDEs that were used in Sections 4.3 and 5.3. Here the w, δ depend on the PDE system under consideration. We also derive an elliptic estimate for distributional solutions with L^2 -regularity to one of the PDEs.

C.1. Weak formulation of boundary value problems in weighted spaces. First, we define corresponding dual spaces.

Definition C.1 (Dual spaces of weighted Sobolev spaces). *Let $(\overline{H}_\delta^w)^*$ denote the space of linear maps $G : \overline{H}_\delta^w \rightarrow \mathbb{R}$ such that there exists a constant $c > 0$ so that*

$$|G(u)| \leq c \|u\|_{\overline{H}_\delta^w} \text{ for all } u \in \overline{H}_\delta^w.$$

Let the norm $\|G\|_{(\overline{H}_\delta^w)^}$ be defined as the smallest $c > 0$ such that the above inequality holds.*

The next lemma shows how weights behave with respect to the dual spaces.

Lemma C.2. *Let $w \geq 0$, $v \in \overline{H}_\delta^0$ and $\alpha \in \mathbb{N}^3$ a multi-index such that $|\alpha| = w$. Denote by $\partial^\alpha v$ the α -th weak derivative of v . Then*

$$\partial^\alpha v \in \left(\overline{H}_{-\delta+w-3}^w\right)^*.$$

Proof. For $u \in \overline{H}_{-\delta+w-3}^w$, it holds that

$$\begin{aligned} \int_{\mathbb{R}^3 \setminus \overline{B_1}} u \partial^\alpha v &= (-1)^{|\alpha|} \int_{\mathbb{R}^3 \setminus \overline{B_1}} \partial^\alpha u v \\ &\leq \|u\|_{\overline{H}_{-\delta+w-3}^w} \|v\|_{\overline{H}_\delta^0}. \end{aligned}$$

This concludes the proof of Lemma C.2. \square

In Sections 4.3 and 5.3, we consider PDEs of the form

$$\begin{cases} \Delta u + \frac{a}{r} \partial_r u + \frac{b}{r^2} u = f & \text{on } \mathbb{R}^3 \setminus \overline{B_1}, \\ u|_{r=1} = 0, \end{cases} \quad (\text{C.1})$$

where $a, b \in \mathbb{R}$ are constants and $u \in \overline{H}_\delta^1, f \in \left(\overline{H}_{-\delta-1}^1\right)^*$.

Note that if $v \in \overline{H}_\delta^1$, then $r^{-1-2\delta}v \in \overline{H}_{-\delta-1}^1$. Therefore, to apply the standard theory of generalised solutions, see for example [16], we consider *weighted* weak formulations after a formal integration by parts of

$$\int_{\mathbb{R}^3 \setminus \overline{B_1}} r^{-2\delta-1} \left(-\Delta u - \frac{a}{r} \partial_r u - \frac{b}{r^2} u \right) v$$

where $u, v \in \overline{H}_\delta^1$. This leads to the following definition.

Definition C.3 (Weak solutions). *Let $f \in \left(\overline{H}_{-\delta-1}^1\right)^*, a, b \in \mathbb{R}$ be given. A function $u \in \overline{H}_\delta^1$ is called weak solution to*

$$\begin{cases} \Delta u + \frac{a}{r} \partial_r u + \frac{b}{r^2} u = f & \text{on } \mathbb{R}^3 \setminus \overline{B_1}, \\ u|_{r=1} = 0, \end{cases} \quad (\text{C.2})$$

if for all $v \in \overline{H}_\delta^1$ it holds that

$$\mathcal{B}_{\delta,a,b}(u, v) = \int_{\mathbb{R}^3 \setminus \overline{B_1}} r^{-1-2\delta} f v,$$

where $\mathcal{B}_{\delta,a,b}(u, v) : \overline{H}_\delta^1 \times \overline{H}_\delta^1 \rightarrow \mathbb{R}$ is the symmetric bilinear form defined by

$$\begin{aligned} \mathcal{B}_{\delta,a,b}(u, v) := & \int_{\mathbb{R}^3 \setminus \overline{B_1}} r^{-2\delta-1} \nabla u \cdot \nabla v - \frac{a+2\delta+1}{2} r^{-2\delta-2} (v \partial_r u - u \partial_r v) \\ & - \int_{\mathbb{R}^3 \setminus \overline{B_1}} (\delta(a+2\delta+1) + b) r^{-2\delta-3} uv. \end{aligned} \quad (\text{C.3})$$

It is left to the reader to verify that $\mathcal{B}_{\delta,a,b}(u, v) : \overline{H}_\delta^1 \times \overline{H}_\delta^1 \rightarrow \mathbb{R}$ is bounded for all $\delta, a, b \in \mathbb{R}$, that is,

$$|\mathcal{B}_{\delta,a,b}(u, v)| \lesssim \|u\|_{\overline{H}_\delta^1} \|v\|_{\overline{H}_\delta^1} \text{ for all } u, v \in \overline{H}_\delta^1.$$

Let $w \geq 2$ be an integer. Introduce three PDEs on $\mathbb{R}^3 \setminus \overline{B_1}$.

- Consider a scalar function $u^{[\geq 2]}$ on $\mathbb{R}^3 \setminus \overline{B_1}$ that verifies

$$\begin{cases} \Delta u^{[\geq 2]} + \frac{4}{r} \partial_r u^{[\geq 2]} + \frac{6}{r^2} u^{[\geq 2]} = f^{[\geq 2]} & \text{on } \mathbb{R}^3 \setminus \overline{B_1}, \\ u^{[\geq 2]}|_{r=1} = 0, \end{cases} \quad (\text{E1})$$

where $f^{[\geq 2]}$ is a given scalar function on $\mathbb{R}^3 \setminus \overline{B_1}$.

- Consider a scalar function $u^{[\geq 2]}$ on $\mathbb{R}^3 \setminus \overline{B_1}$ that verifies

$$\begin{cases} \Delta u^{[\geq 2]} + \frac{1}{r} \partial_r u^{[\geq 2]} - \frac{3}{r^2} u^{[\geq 2]} = f^{[\geq 2]} & \text{on } \mathbb{R}^3 \setminus \overline{B_1}, \\ u^{[\geq 2]}|_{r=1} = 0, \end{cases} \quad (\text{E2})$$

where $f^{[\geq 2]}$ is a given scalar function on $\mathbb{R}^3 \setminus \overline{B_1}$.

- Consider a scalar function $u^{[\geq 1]}$ on $\mathbb{R}^3 \setminus \overline{B_1}$ that verifies

$$\begin{cases} \Delta u^{[\geq 1]} - \frac{1}{2} \Delta u^{[\geq 1]} + \frac{1}{r} \partial_r u^{[\geq 1]} + \frac{1}{r^2} u^{[\geq 1]} = f^{[\geq 1]} & \text{on } \mathbb{R}^3 \setminus \overline{B_1}, \\ u^{[\geq 1]}|_{r=1} = 0, \end{cases} \quad (\text{E3})$$

where $f^{[\geq 1]}$ is a given scalar function on $\mathbb{R}^3 \setminus \overline{B_1}$.

Notice that **(E1)** corresponds to (4.32), **(E2)** to (4.37) and **(E3)** to (5.24).

C.2. Elliptic estimates in \overline{H}_δ^1 . The next proposition shows existence and first elliptic estimates for the above PDEs in weighted Sobolev spaces.

Proposition C.4. *The following holds.*

- Let $f^{[\geq 2]} \in \left(\overline{H}_{1/2}^1\right)^*$. There exists a unique weak solution $u^{[\geq 2]} \in \overline{H}_{-3/2}^1$ to **(E1)** bounded by

$$\|u^{[\geq 2]}\|_{\overline{H}_{-3/2}^1} \lesssim \|f^{[\geq 2]}\|_{\left(\overline{H}_{1/2}^1\right)^*} \quad (\text{C.4})$$

- Let $f^{[\geq 2]} \in \left(\overline{H}_{3/2}^1\right)^*$. There exists a unique weak solution $u^{[\geq 2]} \in \overline{H}_{-5/2}^1$ to **(E2)** bounded by

$$\|u^{[\geq 2]}\|_{\overline{H}_{-5/2}^1} \lesssim \|f^{[\geq 2]}\|_{\left(\overline{H}_{3/2}^1\right)^*} \quad (\text{C.5})$$

- Let $f^{[\geq 1]} \in \left(\overline{H}_{-1/2}^1\right)^*$. There exists a unique weak solution $u^{[\geq 1]} \in \overline{H}_{-1/2}^1$ to **(E3)** bounded by

$$\|u^{[\geq 1]}\|_{\overline{H}_{-1/2}^1} \lesssim \|f^{[\geq 1]}\|_{\left(\overline{H}_{-1/2}^1\right)^*} \quad (\text{C.6})$$

To prove the above proposition, we use the following Poincaré inequality.

Lemma C.5. *Let $n \geq 1$ be an integer. Let the scalar function $u^{[\geq n]} \in C^\infty(\mathbb{R}^3)$. For any $r > 0$ it holds that*

$$\int_{S_r} \frac{(u^{[\geq n]})^2}{r^2} \leq \frac{1}{n(n+1)} \int_{S_r} |\nabla u^{[\geq n]}|^2. \quad (\text{C.7})$$

Proof. Indeed, write

$$\begin{aligned} \int_{S_r} \frac{(u^{[\geq n]})^2}{r^2} &= \sum_{l \geq n} \sum_{m=-l}^l \frac{(u^{(lm)})^2}{r^2}, \\ &= \sum_{l \geq n} \sum_{m=-l}^l \frac{1}{l(l+1)} \frac{l(l+1)}{r^2} (u^{(lm)})^2, \\ &\leq \sum_{l \geq n} \sum_{m=-l}^l \frac{1}{n(n+1)} \frac{l(l+1)}{r^2} (u^{(lm)})^2, \\ &= \frac{1}{n(n+1)} \int_{S_r} |\nabla u^{[\geq n]}|^2, \end{aligned}$$

where we used Lemma 2.33. This proves Lemma C.5. \square

Proof of Proposition C.4. We show for each PDE **(E1)**–**(E3)** that the corresponding bilinear form $\mathcal{B}_{\delta,a,b}$ defined in (C.3) is coercive on the respective weighted space. The a, b corresponding to the PDEs are specified by comparing to (C.1). By the Lax-Milgram Theorem, see for example [16], existence, uniqueness and the claimed estimates follow.

Estimate (C.4). For **(E1)**, $a = 4, b = 6$. We derive the coercivity of $\mathcal{B}_{-3/2,4,6}$ by Lemma C.5 with $n = 2$ as follows,

$$\begin{aligned}
\mathcal{B}_{-3/2,4,6}(u^{[\geq 2]}, u^{[\geq 2]}) &= \int_{\mathbb{R}^3 \setminus \overline{B_1}} r^2 |\nabla u^{[\geq 2]}|^2 - 3 (u^{[\geq 2]})^2 \\
&\geq \int_{\mathbb{R}^3 \setminus \overline{B_1}} r^2 |\nabla u^{[\geq 2]}|^2 - \frac{1}{2} r^2 |\nabla u^{[\geq 2]}|^2, \\
&\geq \frac{1}{2} \int_{\mathbb{R}^3 \setminus \overline{B_1}} r^2 |\nabla u^{[\geq 2]}|^2 \\
&\gtrsim \|u^{[\geq 2]}\|_{\overline{H}_{-3/2}^1}^2,
\end{aligned} \tag{C.8}$$

where we used Lemma C.5 in the second and the last line. This proves (C.4).

Estimate (C.5). For **(E2)**, $a = 1, b = -3$. We estimate from below with Lemma C.5

$$\begin{aligned}
\mathcal{B}_{-5/2,1,-3}(u^{[\geq 2]}, u^{[\geq 2]}) &= \int_{\mathbb{R}^3 \setminus \overline{B_1}} r^4 |\nabla u^{[\geq 2]}|^2 - \frac{9}{2} r^2 (u^{[\geq 2]})^2 \\
&\geq \int_{\mathbb{R}^3 \setminus \overline{B_1}} r^4 |\nabla u^{[\geq 2]}|^2 - \frac{9}{12} r^4 |\nabla u^{[\geq 2]}|^2 \\
&\gtrsim \|u^{[\geq 2]}\|_{\overline{H}_{-5/2}^1}^2.
\end{aligned} \tag{C.9}$$

This proves (C.5).

Estimate (C.6). The symmetric bilinear form $\tilde{\mathcal{B}}$ associated to the weighted weak formulation of **(E3)** in $\overline{H}_{-1/2}^1$ is in fact given by

$$\tilde{\mathcal{B}}(u, v) := \int_{\mathbb{R}^3 \setminus \overline{B_1}} \nabla u \nabla v - \frac{1}{2} \nabla u \nabla v - \frac{1}{2r} (v \partial_r u - u \partial_r v) - \frac{1}{2r^2} uv.$$

Estimate this from below by Lemma C.5

$$\begin{aligned}
\tilde{\mathcal{B}}(u^{[\geq 1]}, u^{[\geq 1]}) &= \int_{\mathbb{R}^3 \setminus \overline{B_1}} |\nabla u^{[\geq 1]}|^2 - \frac{1}{2} |\nabla u^{[\geq 1]}|^2 - \frac{1}{2r^2} (u^{[\geq 1]})^2 \\
&\geq \int_{\mathbb{R}^3 \setminus \overline{B_1}} |\nabla u^{[\geq 1]}|^2 - \frac{1}{2} |\nabla u^{[\geq 1]}|^2 - \frac{1}{4} |\nabla u^{[\geq 1]}|^2 \\
&\gtrsim \|u^{[\geq 1]}\|_{\overline{H}_{-1/2}^1}^2.
\end{aligned} \tag{C.10}$$

This proves (C.6) and hence finishes the proof of Proposition C.4. \square

C.3. Higher elliptic regularity in $H_\delta^w(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{H}_\delta^1$ and \overline{H}_δ^w . In this section, we prove higher elliptic regularity estimates in $H_\delta^w(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{H}_\delta^1$, $w \geq 2$ and \overline{H}_δ^w , for the boundary value problems (E1)-(E3), on the domain $\mathbb{R}^3 \setminus \overline{B_1}$.

Proposition C.6 (Higher regularity in $H_\delta^w(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{H}_\delta^1$). *Let $w \geq 2$ be an integer. The following holds.*

- Let $f^{[\geq 2]} \in H_{-7/2}^{w-2}$. Then the solution $u^{[\geq 2]}$ to (E1) satisfies

$$u^{[\geq 2]} \in H_{-3/2}^w(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{H}_{-3/2}^1$$

and

$$\|u^{[\geq 2]}\|_{H_{-3/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|f^{[\geq 2]}\|_{H_{-7/2}^{w-2}}. \quad (\text{C.11})$$

- Let $f^{[\geq 2]} \in H_{-9/2}^{w-2}$. Then the solution $u^{[\geq 2]}$ to (E2) satisfies

$$u^{[\geq 2]} \in H_{-5/2}^w(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{H}_{-5/2}^1$$

and

$$\|u^{[\geq 2]}\|_{H_{-5/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|f^{[\geq 2]}\|_{H_{-9/2}^{w-2}}. \quad (\text{C.12})$$

- Let $f^{[\geq 1]} \in H_{-5/2}^{w-2}$. Then the solution $u^{[\geq 1]}$ to (E3) satisfies

$$u^{[\geq 1]} \in H_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{H}_{-1/2}^1$$

and

$$\|u^{[\geq 1]}\|_{H_{-1/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|f^{[\geq 1]}\|_{H_{-5/2}^{w-2}}. \quad (\text{C.13})$$

The idea of the proof of Proposition C.6 is to reduce to the following well-known weighted elliptic estimates of [8].

Proposition C.7. *Let $w \geq 2$ be an integer and $\delta \in \mathbb{R}$. Let on \mathbb{R}^3*

$$Lu := a^{ij} \partial_i \partial_j u + b^i \partial_i u + du$$

with coefficients $a^{ij} - A^{ij} \in H_{\delta_2}^{w_2}$, $b^i \in H_{\delta_1}^{w_1}$, $d \in H_{\delta_0}^{w_0}$ with constants

$$w_k \geq k + 1, w_k \geq w - 2, \delta_k < k - 2 \quad \text{for } k = 0, 1, 2,$$

and constants A^{ij} such that there exists $\lambda > 0$ such that

$$A^{ij} \xi_i \xi_j \geq \lambda |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^3.$$

Then there exists a constant $c > 0$ such that for every $u \in H_{loc}^w \cap H_\delta^0$ the following inequality holds

$$\|u\|_{H_\delta^w} \leq c \left(\|Lu\|_{H_{\delta-2}^{w-2}} + \|u\|_{H_\delta^0} \right).$$

The proof of the next lemma is left to the reader.

Lemma C.8. *The operators of (E1)-(E3) satisfy the assumptions of Proposition C.7.*

Proof of Proposition C.6. First, we derive (C.11). Recall (E1), that is,

$$\begin{cases} Lu^{[\geq 2]} := \Delta u^{[\geq 2]} + \frac{4}{r} \partial_r u^{[\geq 2]} + \frac{6}{r^2} u^{[\geq 2]} = f^{[\geq 2]} & \text{on } \mathbb{R}^3 \setminus \overline{B_1}, \\ u^{[\geq 2]}|_{r=1} = 0. \end{cases}$$

The above differential operator L is pointwise elliptic on $\mathbb{R}^3 \setminus \overline{B_1}$ and has smooth coefficients. The Dirichlet boundary data is trivial and by assumption $f^{[\geq 2]} \in H_{-5/2}^{w-2}$. Therefore standard elliptic boundary estimates, see for example Theorem 8.13 in [16], imply that

$$\|u^{[\geq 2]}\|_{H^w(B_2 \setminus \overline{B_1})} \lesssim \|f^{[\geq 2]}\|_{H^{w-2}(B_3 \setminus \overline{B_1})}. \quad (\text{C.14})$$

Use standard Sobolev extension, see for example Theorem 7.25 in [16], to extend the scalar function $u^{[\geq 2]} \in H^w(B_2 \setminus \overline{B_1})$ to $\tilde{u}^{[\geq 2]}$ on B_2 such that

$$\|\tilde{u}^{[\geq 2]}\|_{H^w(B_2)} \lesssim \|u^{[\geq 2]}\|_{H^w(B_2 \setminus \overline{B_1})}.$$

Let $\tilde{\chi} : \mathbb{R}^3 \rightarrow [0, 1]$ be a smooth function such that

$$\tilde{\chi} = \begin{cases} 0 & \text{for } |x| \leq 1/10, \\ 1 & \text{for } |x| \geq 1/2 \end{cases} \quad (\text{C.15})$$

and define the operator \tilde{L} by

$$\tilde{L}\varphi := \Delta\varphi + \frac{4\tilde{\chi}}{r} \partial_r \varphi + \frac{6\tilde{\chi}}{r^2} \varphi \quad \text{on } \mathbb{R}^3$$

for all $\varphi \in C^\infty(\mathbb{R}^3)$. Finally, let

$$\tilde{f}^{[\geq 2]} = \tilde{L}\tilde{u}^{[\geq 2]}.$$

It holds that $\tilde{f}^{[\geq 2]} \in H_{-7/2}^{w-2}$, so that standard interior elliptic estimates, see for example Theorem 8.10 in [16], show that $\tilde{u}^{[\geq 2]} \in H_{loc}^w(\mathbb{R}^3)$.

In summary, it holds that $\tilde{u}^{[\geq 2]} \in \overline{H}^1 \cap H_{loc}^w(\mathbb{R}^3)$ satisfies

$$\tilde{L}\tilde{u}^{[\geq 2]} = \tilde{f}^{[\geq 2]} \quad \text{on } \mathbb{R}^3,$$

where \tilde{L} is a pointwise elliptic operator with smooth coefficients that satisfies the assumptions of Proposition C.7, see also Lemma C.8. Consequently, we can apply Proposition

C.7 to get the weighted estimate

$$\begin{aligned}
\|u^{[\geq 2]}\|_{H_{-3/2}^w(\mathbb{R}^3 \setminus \overline{B_1})} &\leq \|\tilde{u}^{[\geq 2]}\|_{H_{-3/2}^w} \\
&\lesssim \|\tilde{f}^{[\geq 2]}\|_{H_{-7/2}^{w-2}} \\
&\lesssim \|\tilde{u}^{[\geq 2]}\|_{H^w(B_2)} + \|f^{[\geq 2]}\|_{H_{-7/2}^{w-2}} \\
&\lesssim \|u^{[\geq 2]}\|_{H^w(B_2 \setminus \overline{B_1})} + \|f^{[\geq 2]}\|_{H_{-7/2}^{w-2}} \\
&\lesssim \|f^{[\geq 2]}\|_{H_{-7/2}^{w-2}},
\end{aligned}$$

where we used (C.14), the fact that $\tilde{f}^{[\geq 2]} = f^{[\geq 2]}$ on $\mathbb{R}^3 \setminus \overline{B_1}$ and Proposition C.7. This proves (C.11). The estimates (C.12) and (C.13) are proved similarly by using Lemma C.8 and are left to the reader. This finishes the proof of Proposition C.6. \square

Furthermore, we have the following result in \overline{H}_δ^w .

Proposition C.9 (Higher regularity for (E1), (E2), (E3)). *Let $w \geq 2$ be an integer. The following holds.*

- Let $f^{[\geq 2]} \in \overline{H}_{-7/2}^{w-2}$. If the solution $u^{[\geq 2]} \in \overline{H}_{-3/2}^1 \cap H_{-3/2}^2(\mathbb{R}^3 \setminus \overline{B_1})$ to (E1) satisfies

$$\partial_r u^{[\geq 2]}|_{r=1} = 0,$$

then it holds that $u^{[\geq 2]} \in \overline{H}_{-3/2}^w$.

- Let $f^{[\geq 2]} \in \overline{H}_{-9/2}^{w-2}$. If the solution $u^{[\geq 2]} \in \overline{H}_{-5/2}^1 \cap H_{-5/2}^2(\mathbb{R}^3 \setminus \overline{B_1})$ to (E2) satisfies

$$\partial_r u^{[\geq 2]}|_{r=1} = 0,$$

then it holds that $u^{[\geq 2]} \in \overline{H}_{-5/2}^w$.

- Let $w \geq 2$ be an integer. Let $f^{[\geq 1]} \in \overline{H}_{-5/2}^{w-2}$. If the solution $u^{[\geq 1]} \in \overline{H}_{-1/2}^1 \cap H_{-1/2}^2(\mathbb{R}^3 \setminus \overline{B_1})$ to (E3) satisfies

$$\partial_r u^{[\geq 1]}|_{r=1} = 0,$$

then it holds that $u^{[\geq 1]} \in \overline{H}_{-1/2}^w$.

Proof. Proposition C.9 follows by Propositions C.6 and 2.13. Indeed, all necessary normal derivatives on $r = 1$ can be expressed via the equations and shown to vanish. \square

C.4. An elliptic estimate in L^2 . In Section 4.3, we considered the following Dirichlet problem on $\mathbb{R}^3 \setminus \overline{B_1}$ for a scalar function $u^{[\geq 2]} \in H_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})$,

$$\begin{cases} \Delta u^{[\geq 2]} + \frac{1}{r} \partial_r u^{[\geq 2]} - \frac{3}{r^2} u^{[\geq 2]} = \partial_r \left(\text{curl} \left(f_H^{[\geq 2]} \right) \right), \\ u^{[\geq 2]}|_{r=1} = 0, \end{cases} \tag{C.16}$$

where $f_H^{[\geq 2]} \in \mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})$, $w \geq 2$ was a given vectorfield. In the previous sections C.2 and C.3 where this PDE was denoted **(E2)**, we derived elliptic estimates in case $w \geq 3$. In this section we derive estimates for the case $w = 2$.

First, we derive a distributional formulation of (C.16). Let

- $f_H^{[\geq 2]} \in C^\infty(\mathbb{R}^3)$,
- $u \in C^\infty(\mathbb{R}^3)$ be a solution to (C.16),
- $\phi \in C_c^\infty(\mathbb{R}^3)$ such that $\phi|_{r=1} = 0$.

Then, by integrating by parts twice,

$$\begin{aligned} 0 &= \int_{\mathbb{R}^3 \setminus \overline{B_1}} r^4 \left(\Delta u^{[\geq 2]} + \frac{1}{r} \partial_r u^{[\geq 2]} - \frac{3}{r^2} u^{[\geq 2]} - \partial_r \left(\text{curl} \left(f_H^{[\geq 2]} \right) \right) \right) \phi \\ &= \int_{\mathbb{R}^3 \setminus \overline{B_1}} r^4 u^{[\geq 2]} \left(\Delta \phi^{[\geq 2]} + \frac{7}{r} \partial_r \phi^{[\geq 2]} + \frac{12}{r^2} \phi^{[\geq 2]} \right) \\ &\quad - \int_{\mathbb{R}^3 \setminus \overline{B_1}} r^4 f_H^{[\geq 2]} \cdot \left(\frac{6}{r} * (\nabla \phi^{[\geq 2]}) + * (\nabla (\partial_r \phi^{[\geq 2]})) \right), \end{aligned}$$

where here $*(\nabla \phi)_A := \epsilon_{AB} (\nabla \phi)^B$ denotes the Hodge dual of $\nabla \phi$. Here the boundary terms

$$\int_{S_1} \partial_r u \phi, \int_{S_1} u \partial_r \phi, \int_{S_1} u \phi, \int_{S_1} \text{curl} f_H^{[\geq 2]} \phi$$

vanished by the assumptions. The right-hand side still makes sense for

$$\begin{aligned} f_H^{[\geq 2]} &\in \mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1}), \\ u^{[\geq 2]} &\in H_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1}), \\ \phi &\in H_{-5/2}^2(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{H}_{-5/2}^1. \end{aligned}$$

Note the dense inclusion

$$\{\phi \in C_c^\infty(\mathbb{R}^3) : \phi|_{r=1} = 0\} \subset H_{-5/2}^2(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{H}_{-5/2}^1 \cap C^\infty(\mathbb{R}^3 \setminus \overline{B_1}),$$

which is proved by using cut-off functions and left to the reader. This leads to the following definition.

Definition C.10. Let $f_H^{[\geq 2]} \in \mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B_1})$ be a vectorfield. A function $u^{[\geq 2]} \in H_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})$ is called a distributional solution to

$$\begin{cases} \Delta u^{[\geq 2]} + \frac{1}{r} \partial_r u^{[\geq 2]} - \frac{3}{r^2} u^{[\geq 2]} = \partial_r \left(\text{curl} \left(f_H^{[\geq 2]} \right) \right), \\ u^{[\geq 2]}|_{r=1} = 0, \end{cases}$$

if for all $\phi^{[\geq 2]} \in H_{-5/2}^2(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{H}_{-5/2}^1 \cap C^\infty(\mathbb{R}^3 \setminus \overline{B_1})$,

$$\begin{aligned} & \int_{\mathbb{R}^3 \setminus \overline{B_1}} r^4 u^{[\geq 2]} \left(-\Delta \phi^{[\geq 2]} - \frac{7}{r} \partial_r \phi^{[\geq 2]} - \frac{12}{r^2} \phi^{[\geq 2]} \right) \\ &= - \int_{\mathbb{R}^3 \setminus \overline{B_1}} r^4 f_H^{[\geq 2]} \cdot \left(\frac{6}{r} {}^*(\nabla \phi^{[\geq 2]}) + {}^*(\nabla (\partial_r \phi^{[\geq 2]})) \right). \end{aligned} \quad (\text{C.17})$$

Note that this distributional solution is unique in $H_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})$ in view of Lemma C.11 below.

The next lemma is the main result of this section.

Lemma C.11. *Let $f_H^{[\geq 2]} \in \mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})$. Let $u^{[\geq 2]} \in H_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})$ be a distributional solution to (C.16). Then it holds that*

$$\|u^{[\geq 2]}\|_{H_{-5/2}^0} \lesssim \|f_H^{[\geq 2]}\|_{\mathcal{H}_{-5/2}^0}.$$

Proof of Lemma C.11. To prove that $u^{[\geq 2]} \in H_{-5/2}^0 = \overline{H}_{-5/2}^0 = \left(\overline{H}_{-1/2}^0\right)^*$ with

$$\|u^{[\geq 2]}\|_{H_{-5/2}^0} \lesssim \|f_H^{[\geq 2]}\|_{\mathcal{H}_{-5/2}^0},$$

it suffices to show that for all $\varphi^{[\geq 2]} \in C_c^\infty(\mathbb{R}^3 \setminus \overline{B_1})$,

$$\int_{\mathbb{R}^3 \setminus \overline{B_1}} u^{[\geq 2]} \varphi^{[\geq 2]} \lesssim \|f_H^{[\geq 2]}\|_{\mathcal{H}_{-5/2}^0} \|\varphi^{[\geq 2]}\|_{H_{-1/2}^0}.$$

In the following, we will prove that for all $\varphi^{[\geq 2]} \in C_c^\infty(\mathbb{R}^3 \setminus \overline{B_1})$,

$$\int_{\mathbb{R}^3 \setminus \overline{B_1}} r^4 u^{[\geq 2]} \varphi^{[\geq 2]} \lesssim \|f_H^{[\geq 2]}\|_{\mathcal{H}_{-5/2}^0} \|\varphi^{[\geq 2]}\|_{H_{-9/2}^0}, \quad (\text{C.18})$$

which implies the above estimate by the fact that

$$\|r^{-4} f_H^{[\geq 2]}\|_{\mathcal{H}_{-9/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \simeq \|f_H^{[\geq 2]}\|_{\mathcal{H}_{-1/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}$$

It remains to prove (C.18). For given $\varphi^{[\geq 2]} \in C_c^\infty(\mathbb{R}^3 \setminus \overline{B_1})$, let $\Psi^{[\geq 2]}$ be defined as solution to

$$\begin{cases} -\Delta \Psi^{[\geq 2]} - \frac{7}{r} \partial_r \Psi^{[\geq 2]} - \frac{12}{r} \Psi^{[\geq 2]} = \varphi^{[\geq 2]} & \text{on } \mathbb{R}^3 \setminus \overline{B_1}, \\ \Psi^{[\geq 2]}|_{r=1} = 0. \end{cases} \quad (\text{C.19})$$

The operator in (C.19) is the adjoint to **(E2)** with respect to weighted scalar product

$$(u, v) \mapsto \int_{\mathbb{R}^3 \setminus \overline{B_1}} r^4 uv.$$

Therefore (C.19) has the same weak formulation, ellipticity and higher regularity estimates for its generalised solutions as **(E2)**. By Proposition C.6 and standard local interior and boundary elliptic regularity,

$$\Psi \in H_{-5/2}^2(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{H}_{-5/2}^1 \cap C^\infty(\mathbb{R}^3 \setminus \overline{B_1})$$

with

$$\|\Psi^{[\geq 2]}\|_{H_{-5/2}^2(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|\varphi^{[\geq 2]}\|_{H_{-9/2}^0}. \quad (\text{C.20})$$

Plugging now Ψ into (C.17), using (C.20) and (C.19), we get

$$\begin{aligned} \int_{\mathbb{R}^3 \setminus \overline{B_1}} r^4 u^{[\geq 2]} \varphi^{[\geq 2]} &= \int_{\mathbb{R}^3 \setminus \overline{B_1}} r^4 u^{[\geq 2]} \left(-\Delta \Psi^{[\geq 2]} - \frac{7}{r} \partial_r \Psi^{[\geq 2]} - \frac{12}{r} \Psi^{[\geq 2]} \right) \\ &= - \int_{\mathbb{R}^3 \setminus \overline{B_1}} r^4 f_H^{[\geq 2]} \cdot \left(\frac{6}{r} * (\nabla \Psi^{[\geq 2]}) + * (\nabla (\partial_r \Psi^{[\geq 2]})) \right) \\ &\lesssim \|f_H^{[\geq 2]}\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \|\Psi^{[\geq 2]}\|_{H_{-5/2}^2(\mathbb{R}^3 \setminus \overline{B_1})} \\ &\lesssim \|f_H^{[\geq 2]}\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \|\varphi^{[\geq 2]}\|_{H_{-9/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}. \end{aligned}$$

This proves (C.18) and finishes the proof of Lemma C.11. \square

Remark C.12. *The existence of a solution $u^{[\geq 2]} \in H_{-5/2}^0$ to **(E2)** can be deduced from Lemma C.11 by a limit argument. Indeed, it suffices to take $\left(f_H^{[\geq 2]}\right)_n \in C_c^\infty(\mathbb{R}^3 \setminus \overline{B_1})$, $n \in \mathbb{N}$ a sequence such that $\left(f_H^{[\geq 2]}\right)_n \rightarrow f_H^{[\geq 2]}$ in $\mathcal{H}_{-5/2}^0$ as $n \rightarrow \infty$. The corresponding solutions $(u^{[\geq 2]})_n \in \overline{H}_{-5/2}^1$ whose existence is assured by Proposition C.4 will by Lemma C.11 converge to the distributional solution $u^{[\geq 2]}$ in $H_{-5/2}^0$.*

C.5. Estimates to apply Lemma C.2. To apply the above elliptic theory in Section 4.3, the following corollary is used. It follows by the operator analysis in Section 4 and its proof is left to the reader.

Corollary C.13. *The right-hand sides of the PDEs (4.32) and (4.37) can be estimated as follows. We have*

$$\begin{aligned} \left\| \frac{1}{r^3} \partial_r \left(r^3 (\rho_N)^{[\geq 2]} \right) - \operatorname{div} \left(\rho_E^{[\geq 2]} + \zeta_E \right) \right\|_{(\overline{H}_{1/2}^1)^*} &\lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^0} + \|\zeta_E\|_{\overline{\mathcal{H}}_{-5/2}^0}, \\ \left\| \rho_H^{[\geq 2]} + \zeta_H \right\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} &\lesssim \|\rho\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} + \|\zeta_H\|_{\mathcal{H}_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \\ \left\| \partial_r \operatorname{curl} \left(\rho_H^{[\geq 2]} + \zeta_H \right) \right\|_{(\overline{H}_{3/2}^1(\mathbb{R}^3 \setminus \overline{B_1}))^*} &\lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^1(\mathbb{R}^3 \setminus \overline{B_1})} + \|\zeta_H\|_{\overline{\mathcal{H}}_{-5/2}^1(\mathbb{R}^3 \setminus \overline{B_1})}. \end{aligned}$$

Also for $w \geq 3$,

$$\left\| \frac{1}{r^3} \partial_r \left(r^3 (\rho_N)^{[\geq 2]} \right) - \operatorname{div} \left(\rho_E^{[\geq 2]} + \zeta_E \right) \right\|_{\overline{H}_{-7/2}^{w-3}} \lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}} + \|\zeta_E\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}},$$

and for $w \geq 4$,

$$\left\| \partial_r \left(\operatorname{curl} \left(\rho_H^{[\geq 2]} + \zeta_H \right) \right) \right\|_{\overline{H}_{-9/2}^{w-4}} \lesssim \|\rho\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}} + \|\zeta_H\|_{\overline{\mathcal{H}}_{-5/2}^{w-2}}.$$

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